University of California
Los Angeles

Local Operator Spaces and Applications

A dissertation submitted in partial satisfaction
of the requirements for the degree
Doctor of Philosophy in Mathematics

by

Corran James Webster

1997

to Boyd and Di.

Contents

Acknowledgments

First and foremost, it is my pleasure to thank Professor Ed Effros for the advice, assistance and direction he provided as my advisor during this project. I would also like to extend a special thank-you to my colleague and, for a time, fellow student, Soren Winkler, for getting me hooked into the study of operator spaces and for the many fruitful discussions we had while he was at UCLA and via e-mail since then.

I would also like to acknowledge and thank Professor Marc Rieffel, who got the ball rolling back in 1995 in Waterloo; Zhong-Jin Ruan, for many helpful comments, especially concerning compactness arguments for operator spaces; and Professor Masamichi Takesaki for his comments and for the illuminating courses he taught at UCLA. I would also like to thank my fellow students Tony Falcone and Eduard Vaysleb for a number of useful discussions.

Finally, I would like to thank my siblings, Koa, Romily and Vana, for helping to keep my feet on the ground during my journey to this alien shore.

Chapter 1

Introduction

The field which might best be called "quantized functional analysis" was born out of the study of the properties of operator algebras, itself a child of von Neumann's program of "quantizing" mathematics to adapt it to the study of quantum mechanics, inspired by Heisenberg's "matrix mechanics." Murray and von Neumann's early work produced a version of non-commutative integration theory [26]: the basic idea being that spaces of functions associated with the measure space, such as L^∞ are to be replaced by a corresponding space of bounded operators on a Hilbert space, such as $B(H)$ itself. As these spaces were often *-algebras of operators, a rich theory concentrating largely on the algebraic structure developed, and continues to develop, some 60 years on. Moreover the basic program of quantization by replacing functions by operators has led to such developments as Connes' noncommutative geometry [5], and Voiculescu's free probability theory [39], to name a couple of recent developments.

Arveson's 1969 paper [1] on subalgebras of C*-algebras initiated the study of spaces of operators in their own right. The key theorem from this point of view was the fact that there was a version of the Hahn-Banach extension theory which applied to linear spaces of operators on a Hilbert space, the so-called operator spaces. This work was picked up by Effros and Choi, Wittstock, Paulsen and others who proved a number of other results, including a characterization of self-adjoint subspaces and a generalization of the GNS construction. This period of development concluded with Ruan's celebrated representation theorem which provided an axiomatization for operator spaces.

With the key pieces in place, and the important philosophical viewpoint that operator spaces are the "quantized" or "non-commutative" versions of Banach spaces, the theory developed in leaps and bounds driven by the work of Effros and Ruan, Blecher and Paulsen, Sinclair, Smith, Christensen, Kirchberg and Pisier. During the space of a few years many of results of aspects of classical Banach space theory had been generalized to this new setting. This development was facilitated by following the program of Grothendieck, concentrating on the language of tensor products, a very natural concept to workers coming in from operator algebras, rather than the equivalent language of mapping spaces that dominated Banach space theory (and for which operator space versions were also created). Again, after nearly 30 years, this field is vigorous, and it appears that many important concepts in the theory of operator algebras, such as injectivity and exactness, stem from attributes gained from the underlying operator space structure.

That the Hahn-Banach extension theorem is in fact a theorem about convexity was noticed by Wittstock, who was the first to introduce the notion of operator or matrix convexity in his 1979 paper where he extended Arveson's original result to a much fuller generality. His methods, however, were difficult and did not extend easily. Variants and relations of this notion, in particular the C*-convexity worked upon by Loebl and Paulsen [25], Hopenwasser, Moore and Paulsen [19], and Farenick and Morenz [16], were explored and some interesting results discovered, but it was not until some 15 years after Wittstock's original work that Winkler, as part of his doctoral research [42] under Effros, bought Wittstock's convexity back into the fold by providing an axiomatization more in harmony with Ruan's axiomatization for operator spaces, and proceeded to prove a version of the bipolar theorem and, from this, a simplified proof of the Arveson-

Wittstock Hahn-Banach theorem in even greater generality.

One important difference, and a source of much difficulty, between the classical theories in functional analysis and the new quantized versions was that unlike unbounded functions, unbounded operators are not particularly well-behaved. This is of key importance in many applications, in particular in quantum theory where perhaps the most fundamental objects, the observables, are usually unbounded. It is also of importance in non-commutative geometry, as differential operators are usually unbounded, and in the theory of "locally compact" quantum groups, where the known algebraic examples involve elements which correspond to unbounded functions. Classically, spaces of unbounded functions are typically dealt with by looking at them not as elements of normed spaces, but as elements of seminormed spaces. Indeed all the classical examples of interest fall into the category of locally convex topological vector spaces (henceforth called locally convex spaces for brevity).

One might hope that there is a theory of "non-commutative locally convex spaces" analogous to the theory of operator spaces, and that using this theory one may be able to apply this to problems in a number of areas of operator algebra and operator space theory. The development of such a theory is the primary objective of this dissertation.

One natural question which arose during the research was "What are the analogues of compact sets in the operator space context?" Several possible definitions arose, and although one of them was the correct one for the problem at hand (generalizing the Arens-Mackey theorem), the other definitions had some interesting properties. In particular the question as to whether or not they are identical is tied to some of the deepest parts of operator space theory, in particular those areas where the theory diverges from the classical theory of Banach spaces.

This thesis is organized into 4 chapters. The first is this introduction, where we introduce the thesis and some basic notation. In the second chapter we develop a theory of absolute matrix convexity which runs parallel to the work of Winkler and Effros. The key reason for this is that the convex sets that are the unit sets of seminorms are more than just convex, and must, for example, be closed under multiplication by a scalar of absolute value less than 1. We introduce the analogous additional condition to the axioms and prove versions of the bipolar and Hahn-Banach theorem for this sort of matrix convexity. We also introduce the notion of M_∞-convexity where we look at a class of special convex sets living in certain bimodules, an approach which will recur throughout the dissertation.

Having put the foundations in place, in the third and largest chapter we work thoroughly through the theory of local operator spaces, the analogues of locally convex spaces, starting by defining them and the way they can be represented as projective limits of operator spaces. We introduce corresponding definitions for all the key notions in the classical theory: tensor products, bounded sets, topologies on mapping spaces, barrelled spaces, bornological spaces, the Mackey topology and nuclear spaces. Using all these we prove a substantial set of results, the highlights of which include a uniform boundedness principle, a version of the Arens-Mackey duality theory, an exact characterization of matrix nuclear local operator spaces as quantizations of classical nuclear spaces, and the uniqueness of these quantizations, and finally a version of Schwartz's kernel theorem.

Chapter 4 is a deviation from what has gone before. In working on the Arens-Mackey theory it was noticed that there are a number of possible ways of looking at "matrix compact sets," but unlike their classical counterparts, these approaches are not equivalent. We concentrate largely on operator spaces and use one of these definitions to extend Effros and Ruan's operator approximation property, providing an equivalent definition for convergence in the stable point-norm topology and a characterization of the operator approximation property in terms of convergence to matrix compact maps. We conclude the chapter by looking at another definition and showing it is related in the same sort of way to the strong operator approximation property. We show that if these two notions of compactness agree, then the two operator approximation property agree. In particular this implies, through the deep work of Kirchberg, that if these two notions agree for a C*-algebra, then operator approximation property is equivalent to exactness. We then investigate a condition which guarantees that the two definitions agree.

Throughout, especially in the first two chapters, we include examples to indicate the applications of this material to other areas.

1.1 Standard Notation and Conventions

Throughout we will assume that the reader has some familiarity with both the theory of operator spaces and the theory of locally convex topological vector spaces, and that they are completely familiar with the theory of C*-algebras and von Neumann algebras. We will often quickly describe the situation in the operator or locally convex space case before going on to adapt it to the situation we are interested in. In the case of locally convex spaces there are many texts available on the subject, such as [4, 18, 33, 38, 44]. The theory of operator algebras is well-documented in such references as [21, 17, 28, 37]. There are no texts on operator spaces which present the current state of the art, although Effros and Ruan are working on a book [7] at present which includes most of the necessary background, and Paulsen's book [27] discusses most of what was known prior to Ruan's work. Articles which a reader might find useful include [1, 43, 34, 8, 2, 3, 31, 30, 32].

Where there is no conflict with the terminology, we may use the adjectives *matrix*, *completely* and *operator* interchangeably. The two most notable dangers with this policy are that an unrelated notion of a completely continuous map already exists in the literature, and that in the discussion of compactness in Chapter 4 we mean distinct conditions by *operator compact*, *matrix compact* and *completely compact*. As result, I have tried to be as consistent as possible.

Given a topological space X we denote the continuous functions, the bounded continuous functions, the functions vanishing at infinity and the functions vanishing off compact sets by $C(X)$, $C_b(X)$, $C_\infty(X)$ and $C_c(X)$ respectively. If X is a manifold, then we replace C by C^k to denote the k-times continuously differentiable functions with the corresponding properties.

We denote the sequences vanishing at 0, the sequences which are 0 for all but finitely many values, the bounded sequences, absolutely summable sequences and square summable sequences of complex numbers as c_0 and c_c, ℓ^∞, ℓ^1 and ℓ^2 respectively. We will denote the standard basis in these spaces by $\{e_k\}$, where e_k is the sequence which is zero everywhere but the kth element.

As is usual, given a Hilbert space H, we let $\mathcal{B}(H)$, $\mathcal{K}(H)$, $\mathcal{T}(H)$, $\mathcal{HS}(H)$ and $\mathcal{F}(H)$ be the bounded, compact, trace-class, Hilbert-Schmidt and finite rank operators on H respectively. When H is ℓ^2, we will often simply write $\mathcal{B}$, $\mathcal{K}$, $\mathcal{T}$, $\mathcal{HS}$ and $\mathcal{F}$. If H is $\mathbb{C}^n$, we will write M_n, $\mathcal{T}_n$ and $\mathcal{HS}_n$ for the bounded, trace-class and Hilbert-Schmidt respectively. Given a basis $\{\xi_\nu\}_{\nu\in\Lambda}$ for H, we will let $e_{\mu,\nu}$ be the partial isometry that takes $\operatorname{span}\{\xi_\nu\}$ to $\operatorname{span}\{\xi_\mu\}$. If $\Gamma \subseteq \Lambda$ we denote by p_Γ the projection from H to $\operatorname{span}\{\xi_\nu : \nu \in \Gamma\}$. If $\Lambda = \mathbb{N}$, then we let $p_n = p_{\{1,\ldots,n\}}$.

In a number of places we will have to deal with arithmetic on the extended positive real numbers $[0,\infty]$. In this case we will use the standard "zero wins" convention for multiplication, i.e.

$$\lambda \cdot \infty = \begin{cases} 0 & \text{if } \lambda = 0, \\ \infty & \text{otherwise.} \end{cases}$$

1.2 Matrix Conventions

We now turn to the notational conventions for working with matrix spaces. Unless otherwise noted, all vector spaces are over the complex numbers. Given vector spaces V and W, we denote by $\mathcal{L}(V,W)$ the space of all linear maps from V to W.

Given two index sets I and J, and a vector space V, we denote by $M_{I,J}(V)$ the vector space of I by J matrices $[v_{i,j}]$, where $i \in I$, $j \in J$, $v_{i,j} \in V$ and all but finitely many of the $v_{i,j}$ are 0. For convenience, we will write $M_I(V)$ for $M_{I,I}(V)$, $M_{I,J}$ for $M_{I,J}(\mathbb{C})$ and allow a natural number n to replace the set $I = \{1,\ldots,n\}$, and ∞ to replace $\mathbb{N}$. The reader should be careful not to confuse $M_{n\times m}(V)$, which are matrices of the form $[v_{(i,j),(k,l)}]$ where (i,j), $(k,l) \in \{1,\ldots,n\} \times \{1,\ldots,m\}$, with $M_{n,m}(V)$, which are matrices of the form $[v_{i,j}]$ where $i \in \{1,\ldots,n\}$ and $j \in \{1,\ldots,m\}$.

Given V, W vector spaces, the algebraic tensor product

$$M_{I,J}(V) \otimes M_{K,L}(W)$$

is isomorphic to $M_{I\times K,J\times L}(V \otimes W)$.

If $I \subset K$ and $J \subset L$ we have standard inclusion maps from $M_{I,J}(V)$ to $M_{K,L}(V)$ given by

$$v \mapsto \begin{bmatrix} v & 0 \\ 0 & 0 \end{bmatrix}.$$

Furthermore, we note that with these inclusions, we have that as vector spaces

$$M_{I,J}(V) = \varinjlim M_{I',J'}(V)$$

where the injective limit is taken over finite subsets $I' \subseteq I$ and $J' \subseteq J$.

If V has additional properties beyond being a vector space, then we can extend these to the matrices over V. If V is a *-vector space then we define an antilinear map

$$M_{I,J}(V) \to M_{J,I}(V) : v \to v^* = [v^*_{j,i}].$$

This map is a *-operation for $M_I(V)$, making it a *-vector space. If V is an algebra, then given matrices $v \in M_{I,J}(V)$ and $w \in M_{J,K}(V)$ we define their product by the *matrix product*

$$vw = \left[\sum_{j\in I} v_{i,j} w_{j,k}\right] \in M_{I,K}(V).$$

In particular, this makes $M_I(V)$ an algebra. More generally, if V is a left A-module for some algebra A, then $M_{I,J}(A)$ acts on $M_{J,K}(V)$ via

$$av = \left[\sum_{j\in I} a_{i,j} v_{j,k}\right] \in M_{I,K}(V).$$

Similarly, if V is a right A-module, then $M_{J,K}(A)$ acts from the right on $M_{I,J}(V)$ in the obvious way. In particular, this means that $M_{I,J}$ acts on $M_{J,K}(V)$ and $M_{K,I}(V)$, for any V, from the left and right respectively.

There is a novel operation on these matrix spaces, that of the *direct sum* of two matrices $v \in M_I(V)$, $w \in M_J(V)$, which is

$$v \oplus w = \begin{bmatrix} v & 0 \\ 0 & w \end{bmatrix} \in M_{I\sqcup J}(V)$$

and for convenience, we will identify $\{1, \ldots, n\} \sqcup \{1, \ldots, m\}$ with $n + m$.

Given a map $\varphi : V \to W$ between two vector spaces we define $\varphi_{I,J} : M_{I,J}(V) \to \varphi_{I,J}(W)$ by

$$\varphi_{I,J}(v) = [\varphi(v_{i,j})].$$

As before we will write φ_I for $\varphi_{I,I}$.

We identify $M_{I,J}(\mathcal{L}(V,W))$ with $\mathcal{L}(V, M_{I,J}(W))$ by mapping the matrix $[\varphi_{i,j}]$ to the function $v \mapsto [\varphi_{i,j}(v)]$.

Given V a vector spaces, $\sigma \in \mathcal{L}(M_I, M_K) \cong M_{I\times K}$, $v \in M_I(V)$, then we define

$$\sigma \cdot_I v = \left[\sum_{i,j\in I} \sigma_{(k,i),(l,j)} v_{i,j}\right]_{k,l\in K}.$$

Alternatively, if we identify $M_I(V)$ with $M_I \otimes V$, we can think of this as

$$\sigma \cdot_I v = (\sigma \otimes \mathrm{id})(v).$$

A (non-degenerate) *pairing* of two vector spaces V and W is a bilinear function

$$\langle \cdot, \cdot \rangle : V \times W \to \mathbb{C}$$

such that if $\langle v, w\rangle = 0$ for all $v \in V$, then $w = 0$; and if $\langle v, w\rangle = 0$ for all $w \in W$, then $v = 0$. So each element $v \in V$ (respectively $w \in W$) determines a linear functional $v : W \to \mathbb{C}$ (respectively $w : V \to \mathbb{C}$) by

$$v(w) = w(v) = \langle v, w\rangle.$$

Given such a pairing we get a (scalar) pairing of $M_{I,J}(V)$ and $M_{I,J}(W)$ given by

$$\langle v, w\rangle = \sum_{i\in I, j\in J} \langle v_{i,j}, w_{i,j}\rangle.$$

However, we also get a *matrix pairing* of $M_{I,J}(V)$ and $M_{K,L}(V)$ which is a map

$$\langle\langle\cdot,\cdot\rangle\rangle : M_{I,J}(V) \times M_{K,L}(W) \to M_{I\times K, J\times L}$$

where the $(i, k), (j, l)$-th entry of $\langle\langle v, w\rangle\rangle$ given by $\langle v_{i,j}, w_{k,l}\rangle$, or equivalently

$$\langle\langle v, w\rangle\rangle = v_{K,L}(w) = w_{I,J}(v).$$

We can extend this to arbitrary bilinear maps $\varphi : V \times W \to X$, we define

$$\varphi_{I,K;J,L}(v, w) : M_{I,K}(V) \times M_{J,L}(W) \to M_{I\times K, J\times L}(X)$$

where the $(i, k), (j, l)$-th entry of $\varphi_{I,K;J,L}(v, w)$ is $\varphi(v_{i,k}, w_{j,l})$. If $I = K$ and $J = L$ the we will write $\varphi_{I;J}$ for $\varphi_{I,K;J,L}$.

We define a *matrix set* $\boldsymbol{K} = (K_n)$ in a vector space V to be a family of sets $K_n \subseteq M_n(V)$ for $n \in \mathbb{N}$. We call K_n the n-th *level* of $\boldsymbol{K}$. If V is a vector space we will write $\boldsymbol{V}$ for the matrix set $(M_n(V))$. We say that one matrix set $\boldsymbol{K}$ is a subset of another $\boldsymbol{L}$ if $K_n \subset L_n$ for all $n \in \mathbb{N}$, and define intersection and union of matrix sets by taking intersections, or unions respectively, at each level. If V is a topological vector space, then we say $\boldsymbol{K}$ is *closed* if it is closed at each level, and we define *open* matrix sets analogously. If $\varphi : V \to W$, then given $\boldsymbol{K} \subseteq \boldsymbol{V}$ we get the matrix image of $\boldsymbol{K}$,

$$\boldsymbol{\varphi}(\boldsymbol{K}) = (\varphi_n(K_n)) \subseteq \boldsymbol{W},$$

and given $\boldsymbol{L} \subseteq \boldsymbol{W}$ we have matrix inverse image

$$\boldsymbol{\varphi}^{-\mathbf{1}}(\boldsymbol{L}) = (\varphi^{-1}(L_n)) \subseteq \boldsymbol{V}.$$

Chapter 2

Matrix Convexity

This chapter is preliminary to the main part of the dissertation. Our objective is to develop a theory analogous to the theory of locally convex spaces in the context of operator spaces. To do this we need to know exactly what we mean by a "convex set," or more specifically, an "absolutely convex set," since in the theory of locally convex spaces, the unit sets of seminorms satisfy the additional properties that they must be closed under multiplications by scalars of absolute value less than or equal to 1. This condition is sometimes referred to the literature as "circled" or "balanced" convexity.

We first define matrix convex sets, a definition which should be natural to those who have seen either the theory of operator spaces or the work of Effros and Winkler. We look at the corresponding notion of matrix gauges, and specialize to look at matrix seminorms. We then move on to prove a version of the bipolar theorem, very closely related to the version due to Effros and Ruan, but adapted to allow for the possibly infinite or zero values that matrix gauge may take, as compared to a matrix norm. We then derive a version of the Hahn-Banach theorem from the version of Effros and Winkler. We conclude with an alternative, but equivalent, way of looking at matrix convexity in terms of M_∞-bimodules.

2.1 Matrix Convexity and Matrix Gauges

Matrix convexity was first introduced by Wittstock [43], who defined a matrix set $\boldsymbol{K}$ to be *matrix convex* if given $v_i \in K_{n_i}$ and $\alpha_i \in M_{n_i,n}$ with $\sum_i \alpha_i^* \alpha_i = I_n$ then

$$\sum_i \alpha_i^* v_i \alpha_i \in K_n.$$

In [15], Effros and Winkler observed that this was equivalent to saying that

(MC1)	$v \oplus w \in K_{m+n}$	for all $v \in K_m$, $w \in K_n$
(MC2)	$\alpha^* v \alpha \in K_n$	for all $v \in K_m$, $\alpha \in M_{m,n}$ such that $\alpha^*\alpha = I_m$.

Note that under this definition the completely positive cones of the operator systems, and the matrix unit balls of operator spaces are both matrix convex.

Recall that the convex sets which form neighborhood bases of 0 of locally convex vector spaces satisfy a stronger condition than convexity, that of absolute convexity, i.e. that if $v_i \in K$ and $\alpha_i \in \mathbb{C}$ such that $\sum |\alpha_i| \leq 1$, then $\sum \alpha_i v_i \in K$. Observing that, in particular, the unit ball of a matrix normed vector space should be absolutely convex, and comparing with the definition of an operator space, we are led to the following definition.

Definition 2.1.1
A matrix set $\boldsymbol{K}$ *is* absolutely matrix convex *if it satisfies*

(AMC1)	$v \oplus w \in K_{m+n}$	*for all* $v \in K_m$, $w \in K_n$
(AMC2)	$\alpha^* v \beta \in K_n$	*for all* $v \in K_m$, $\alpha,\ \beta \in M_{m,n}$ *such that* $\|\alpha\|,\ \|\beta\| \leq 1$.

Classically, absolutely convex neighborhoods of 0 are the unit sets of seminorms. Analogously to [15] we need to generalize this slightly. A *gauge* on a vector space V is a function

$$\gamma : V \to [0,\infty]$$

which satisfies

(G1)	$\gamma(v+w) \leq \gamma(v) + \gamma(w)$	for all $v,\ w \in V$
(G2)	$\gamma(\alpha v) = \|\alpha\|\gamma(v)$	for all $v \in V$, $\alpha \in \mathbb{C}$.

where multiplication of numbers in $[0,\infty]$ follows the "zero wins" convention. Note that this is the absolutely convex analogue of the definition in [15]; should we need to distinguish between the two we will refer to gauges which satisfy (G1) and (G2) as *absolute gauges*.

We say that a gauge p is *faithful* if $p(v) = 0$ implies $v = 0$ and we call a gauge ρ a *seminorm* if $\rho(v) < \infty$ for all $v \in V$. A faithful seminorm is a *norm*. A faithful gauge determines a norm on the subspace

$${}_pV = \{v \in V : p(v) < \infty\}$$

while a seminorm determines a norm on the quotient space

$$V_\rho = V/N_\rho = V/\{v \in V : \rho(v) = 0\}. \tag{2.1}$$

Given a gauge γ, the *unit set*

$$B^\gamma = \{v \in V : \gamma(v) \leq 1\}$$

is *absolutely convex*, that is, given $v_i \in B^\gamma$ and $\alpha_i \in \mathbb{C}$ for $i = 1, \ldots, n$ such that $\sum_{i=1}^n |\alpha_i| \leq 1$, then the *absolutely convex combination* $\sum_{i=1}^n \alpha_i v_i$ lies in B^γ also. Conversely, given any absolutely convex subset B of a vector space V, we get the *Minkowski gauge* γ^B by letting

$$\gamma^B(v) = \inf\{\lambda > 0 : v \in \lambda B\}.$$

Definition 2.1.2
Given a vector space V*, a* matrix gauge $\boldsymbol{\gamma}$ *on* V *is a collection of gauges*

$$\boldsymbol{\gamma} = (\gamma_n : M_n(V) \to [0,\infty])$$

such that $\boldsymbol{\gamma}$ *satisfies*

(AMG1)	$\gamma_{n+m}(v \oplus w) = \max\{\gamma_n(v), \gamma_m(w)\}$	*for all* $v \in M_n(V)$, $w \in M_m(V)$,
(AMG2)	$\gamma_m(\alpha v \beta) = \|\alpha\|\,\gamma_n(v)\,\|\beta\|$	*for all* $v \in M_n(V)$, $\alpha \in M_{m,n}$, $\beta \in M_{n,m}$.

Again, this is the absolutely matrix convex version of a gauge. If there is any confusion between this definition, and that in Effros and Winkler [15], we will say that a matrix gauge which satisfies (AMG1) and (AMG2) is an *absolute* matrix gauge. We note that it suffices to check the following weaker version of (AMG1).

Lemma 2.1.1
Given a vector space V, and a collection of gauges

$$\boldsymbol{\gamma} = (\gamma_n : M_n(V) \to [0, \infty])$$

such that $\boldsymbol{\gamma}$ satisfies

(AMG1') $\quad \gamma_{n+m}(v \oplus w) \leq \max\{\gamma_n(v), \gamma_m(w)\}$ *for all* $v \in M_n(V)$, $w \in M_m(V)$,

(AMG2') $\quad \gamma_m(\alpha v \beta) = \|\alpha\|\, \gamma_n(v)\, \|\beta\|$ *for all* $v \in M_n(V)$, $\alpha \in M_{m,n}$, $\beta \in M_{n,m}$.

then $\boldsymbol{\gamma}$ is a matrix gauge.

Proof :
Let $v \in M_n(V)$, $w \in M_m(V)$, then by (AMG2)

$$\gamma_n(v) = \gamma(\alpha(v \oplus w)\beta) \leq \gamma_{n+m}(v \oplus w)$$

where

$$\alpha = \begin{bmatrix} I_n & 0 \end{bmatrix}, \ \beta = \begin{bmatrix} I_n \\ 0 \end{bmatrix}$$

and similarly $\gamma_m(w) \leq \gamma_{n+m}(v \oplus w)$ so that

$$\max\{\gamma_n(v), \gamma_m(w)\} \leq \gamma_{n+m}(v \oplus w) \leq \max\{\gamma_n(v), \gamma_m(w)\}$$

by (AMG1'). So $\boldsymbol{\gamma}$ satisfies (AMG1). □

It is clear from the way that we have defined things that the unit matrix set of a matrix gauge $\boldsymbol{\gamma}$ is matrix convex. Indeed the matrix set

$$(\{v \in M_n(V) : \gamma_n(v) \leq \lambda\})$$

for any $\lambda \in [0, \infty]$ is matrix convex. Conversely, if we are given a matrix convex set $\boldsymbol{K} = (K_n)$ we have a corresponding *Minkowski matrix gauge* $\boldsymbol{\gamma}^{\boldsymbol{K}}$ of $\boldsymbol{K}$ where $\gamma_n^{\boldsymbol{K}}$ is given by the Minkowski gauge γ^{K_n} of the n-th level of $\boldsymbol{K}$.

If we have a matrix gauge $\boldsymbol{\gamma}$ on a vector space V and a matrix $v = [v_{i,j}] \in M_n(V)$, then

$$\gamma_1(v_{i,j}) \leq \gamma_1(e_{1,i} v e_{1,j}^*) \leq \gamma_n(v) = \gamma_n(\sum_{i,j} e_{1,i}^* v e_{1,j}) \leq n^2 \max\{\gamma_1(v_{i,j})\}. \tag{2.2}$$

Hence if $\boldsymbol{\rho}$ is a matrix gauge and ρ_1 is faithful, then so are all the ρ_n. In this case we call the matrix gauge *faithful*. If $\boldsymbol{\rho}$ is such that ρ_1 is a seminorm, then the above implies that each ρ_n is also a seminorm, and we call such a matrix gauge a *matrix seminorm*. A matrix gauge which is both faithful and a matrix seminorm is called a *matrix norm*.

If $\boldsymbol{\rho}$ is a matrix seminorm on V, we let

$$N_{\boldsymbol{\rho}} = \{v \in V : \rho_1(v) = 0\}$$

be the kernel of $\boldsymbol{\rho}$. Then

$$V_{\boldsymbol{\rho}} = V / N_{\boldsymbol{\rho}} \tag{2.3}$$

has a matrix norm $(\| \cdot \|_n^{\boldsymbol{\rho}})$ determined by $\boldsymbol{\rho}$ via

$$\|\pi_{\boldsymbol{\rho}}(v)\|_n^{\boldsymbol{\rho}} = \rho_n(v)$$

where $\pi_{\boldsymbol{\rho}}$ is the quotient map. The same construction will give a faithful matrix gauge from an arbitrary matrix gauge. Analogously, if $\boldsymbol{\rho}$ were faithful, then if we restrict $\boldsymbol{\rho}$ to the subspace

$$_{\boldsymbol{\rho}}V = \{v \in V : \rho_1(v) < \infty\}$$

then we also get a matrix norm. If we were to start with an arbitrary matrix gauge, we would simply get a matrix seminorm.

If we are given two matrix gauges $\boldsymbol{\gamma}$ and $\boldsymbol{\rho}$, then it is easy to see that

- $\boldsymbol{\gamma} + \boldsymbol{\rho}$, where $(\gamma + \rho)_n = \gamma_n + \rho_n$,
- $\max\{\boldsymbol{\gamma}, \boldsymbol{\rho}\}$, where $\max\{\gamma, \rho\}_n = \max\{\gamma_n, \rho_n\}$,
- $\lambda\boldsymbol{\gamma}$, for any $\lambda \in [0, \infty]$,

are all matrix gauges.

Example 2.1.1
An (abstract) operator space is a vector space V with a family of norms $(\|\cdot\|_n : M_n(V) \to [0, \infty))_{n\in\mathbb{N}}$ which satisfy

(MN1) $\quad \|v \oplus w\|_{n+m} = \max\{\|v\|_n, \|w\|_m\} \quad$ for all $v \in M_n(V)$, $w \in M_m(V)$,

(MN2) $\quad \|\alpha v \beta\|_m = \|\alpha\| \, \|v\|_n \, \|\beta\| \quad$ for all $v \in M_n(V)$, $\alpha \in M_{m,n}$, $\beta \in M_{n,m}$.

We will assume that V is complete at the first level (and so is complete at all levels). Clearly the family of norms give an absolutely matrix convex gauge on V, indeed $(\|\cdot\|_n)$ is a matrix norm.

Any linear subspace of $\mathcal{B}(H)$ is an operator space with the norms $\|\cdot\|_n$ inherited from $\mathcal{B}(H^n)$. Conversely Ruan [34] showed that any abstract operator space is completely isometrically isomorphic to a linear subspace W of $\mathcal{B}(H)$ for some Hilbert space H. That is to say, there is a map $\varphi : V \to W$ such that the maps φ_n are all isometries.

A number of special operator spaces will be of some importance:

- If A is a C*-algebra, then so are the algebras $M_n(A) = M_n \overline{\otimes} A$. In fact we get a matrix norm from the norms on each of the $M_n(A)$. Moreover, this matrix norm is exactly the matrix norm that we get regarding A as a subspace of $\mathcal{B}(H)$ where A is represented faithfully on H.
- If H is a Hilbert space, we can regard it as $\mathcal{B}(\mathbb{C}, H)$ which is a subspace of $\mathcal{B}(H)$. We call the operator space we get this way the column Hilbert space H_c.
- If instead we regard H as $\mathcal{B}(H, \mathbb{C}) \subseteq \mathcal{B}(H)$. We call the operator space constructed this way the row Hilbert space H_r.
- We will see in the following section that any Banach space V has a minimal and a maximal matrix norm for which the norm on the first level is the norm on V.
- If V and W are operator spaces, then we say a linear map $\varphi : V \to W$ is *completely bounded* if the norm
$$\|\varphi\|_{cb} = \sup_{n\in\mathbb{N}} \|\varphi_n\|$$
is finite, where the norm of φ_n is the one it attains as an element of $\mathcal{B}(V, M_n(W))$. We denote the space of all completely bounded maps from V to W by $\mathcal{CB}(V, W)$. Then by since $M_n(\mathcal{CB}(V, W))$ can be identified with $\mathcal{CB}(V, M_n(W))$ we can give $\mathcal{CB}(V, W)$ an operator space structure.
- In particular, if $W = \mathbb{C}$ with the obvious operator space structure, then $\mathcal{CB}(V, \mathbb{C}) = V^* = \mathcal{B}(V, \mathbb{C})$ as Banach spaces, so we have an operator space structure on the dual of V.

Example 2.1.2
Consider the non-commutative torus A_θ, the C*-algebra generated by unitaries u and v with commutation relation

$$vu = e^{2\pi i\theta}uv.$$

We have natural derivations δ_u, δ_v which act via

$$\delta_u(u^m v^n) = 2\pi i m u^m v^n,$$

and

$$\delta_v(u^m v^n) = 2\pi i n u^m v^n.$$

We observe that δ_u and δ_v commute, so we define $D^{k,l} = \delta_u^k \circ \delta_v^l$. Then we have matrix gauges $\boldsymbol{\rho^{k,l}}$ where

$$\rho_n^{k,l}(v) = ||D_n^{k,l}(v)||$$

(or ∞ if $D^{k,l}$ is undefined for some $v_{i,j}$) where the norm is the norm on $M_n \overline{\otimes} A_\theta$. In fact, for applications it is often more convenient to work with the matrix gauges $\boldsymbol{\gamma^i}$ given by the gauges

$$\gamma_n^i(v) = \sum_{k+l\le i} \frac{\rho_n^{k,l}(v)}{k!l!}$$

since these are better behaved with respect to the multiplication.

Example 2.1.3
The previous example can be generalized to the situation where we have a Lie group G acting via a strongly continuous action σ on an operator space V. Let $\mathfrak{g}$ be the Lie algebra of G, and let $x_1, \ldots, x_q$ be a basis for $\mathfrak{g}$. Then for each multiindex $\mu \in \mathbb{N}^q$, we have operators

$$D^\mu(v) = \left(\frac{d}{dt_1}\right)^{\mu_1} \cdots \left(\frac{d}{dt_q}\right)^{\mu_q} \sigma_{\exp(t_1x_1)\ldots\exp(t_qx_q)}(v)\bigg|_{t_1=0,\ldots,t_q=0}$$

and so we have matrix gauges $\boldsymbol{\rho^\mu}$ given by

$$\rho_n^\mu(v) = ||D_n^\mu(v)||_n$$

(or ∞ if D^μ is undefined for some $v_{i,j}$), where the norms $||\cdot||_n$ are those from the operator space V. Again, for applications, it will often be more convenient to work with the operator gauges $\boldsymbol{\gamma^k}$ given by

$$\gamma_n^k(v) = \sum_{|\mu|\le k} \frac{\rho_n^\mu(v)}{\mu!}$$

where $\mu! = \mu_1!\mu_2!\ldots\mu_q!$.

Example 2.1.4
We can generalize Example 2.1.2 in another way. Let a be an unbounded linear operator on an operator space V. Then we have a matrix gauge $\boldsymbol{\gamma^a}$ on V given by

$$\gamma_n^a(v) = \begin{cases} ||a_n(v)||_n & \text{if } v_{i,j} \in \mathcal{D}a, \text{ for all } 1 \le i,j \le n, \\ \infty & \text{otherwise.} \end{cases}$$

where $||\cdot||_n$ is the norm on the n-th level of V.

2.2 The Bipolar Theorem

If V is a locally convex space, a gauge γ on V is *lower semicontinuous* if the corresponding unit set B^γ is closed. The mappings $\gamma \mapsto B^\gamma$ and $B \mapsto \gamma^B$ give a one-to-one correspondence between lower semicontinuous gauges and closed absolutely convex subsets $B \subset V$.

Two locally convex spaces V and W are *in duality* if there is a pairing $\langle\cdot,\cdot\rangle$ between V and W such that all the continuous functionals on V are given by elements of W and vice-versa.

Given a dual pair of locally convex vector spaces V and W, we recall that the *absolute polar* of a subset D of V is defined by

$$D^\circ = \{w \in W : |\langle v, w\rangle| \leq 1, \ \forall v \in D\}.$$

Classically, the bipolar theorem says that $D^{\circ\circ}$ is the smallest weakly closed absolutely convex set containing D.

Given a locally convex vector space V, the canonical topology on $M_{n,m}(V)$ is simply the product topology, regarding $M_{n,m}(V)$ as V^{mn} (which are clearly isomorphic as vector spaces).

If V and W are in duality, then we observe that $M_n(V)$ and $M_n(W)$ are in duality using the scalar pairing. Furthermore we note that the matrix pairing

$$\langle\langle\cdot,\cdot\rangle\rangle : V \times M_n(W) \to M_n$$

determines all the continuous mappings $\varphi : V \to M_n$, that is

$$M_n(W) \cong \mathcal{C}(V, M_n). \tag{2.4}$$

Given a dual pair of locally convex vector spaces V and W and a matrix set $\boldsymbol{D} \subseteq \boldsymbol{V}$, then we define the *(absolute) matrix polar* of $\boldsymbol{D}$ to be the matrix set $\boldsymbol{D}^{\circledcirc}$ given by

$$D_n^{\circledcirc} = \{w \in M_n(W) : \|\langle\langle v, w\rangle\rangle\| \leq 1, \ \forall v \in D_r, \ r \in \mathbb{N}\}.$$

We immediately note that if $\boldsymbol{D}$ is matrix convex

$$(D^{\circledcirc})_1 = (D_1)^\circ$$

since for any $v \in D_r$, and arbitrary unit vectors $\zeta, \xi \in \mathbb{C}^r$, we have that

$$\langle\, \langle\langle w, v\rangle\rangle\zeta | \xi\rangle = \langle w, \zeta^* v \xi\rangle$$

and so if $w \in (D_1)^\circ$ then $w \in D_1^{\circledcirc}$. The other direction of the inclusion is trivial, and true even if $\boldsymbol{D}$ is not matrix convex.

Example 2.2.1
Let V be an operator space, $\boldsymbol{B}$ its matrix unit ball, and $V^* = \mathcal{CB}(V, \mathbb{C})$ the dual operator space of V. Then the absolute matrix polar of $\boldsymbol{B}$ is seen by inspection of the definition to be the matrix unit ball of V^*.

We now have a bipolar theorem which is a variation on that in [11], modified to allow for arbitrary gauges. In this it uses techniques first used in the bipolar theorem in [15]. Although the theorem is important in what will follow, the proof is lengthy and technical, but standard, and so the reader may prefer to skip it on the first reading.

Proposition 2.2.1
Suppose that V and W are dual locally convex spaces. Given any matrix set $\boldsymbol{D} \subseteq \boldsymbol{V}$, we have that $\boldsymbol{D}^{\circledcirc\circledcirc}$ is the smallest weakly closed absolutely matrix convex set containing $\boldsymbol{D}$.

Proof :

It suffices to prove that if $\boldsymbol{D}$ is absolutely matrix convex and weakly closed, then $\boldsymbol{D}^{\circledcirc\circledcirc} = \boldsymbol{D}$. Thus we must prove that if $v_0 \notin D_n$, then $v_0 \notin D_n^{\circledcirc\circledcirc}$. To do this we will show that there exists a $w_0 \in M_n(W)$ for which $w_0 \in D_n^{\circledcirc}$, i.e.,

$$\|\langle\langle v, w_0\rangle\rangle\| \leq 1,$$

for all $v \in D_r$ and $r \in \mathbb{N}$, but

$$\|\langle\langle v_0, w_0\rangle\rangle\| > 1$$

Letting γ be the matrix gauge of $\boldsymbol{D}$ and using the identification (2.4), it suffices to find a mapping $\varphi : V \to M_n$ for which

$$\|\varphi_r(v)\| \leq 1 \tag{2.5}$$

for all $v \in D_r$ and $r \in \mathbb{N}$, but

$$\|\varphi_n(v_0)\| > 1. \tag{2.6}$$

Using the classical bipolar theorem, we have that $v_0 \notin D_n^{\circ\circ}$, i.e., we may find a functional $\psi \in M_n(V)'$ for which

$$|\psi|_{D_n}| \leq 1 < |\psi(v_0)|\,. \tag{2.7}$$

As in [15] we will extract the desired matrix valued function φ from a perturbation of ψ.

Our argument depends on a simple convexity result. Let us suppose that $\mathcal{E}$ is a cone of real continuous affine functions on a compact convex subset K of a topological vector space E, and that for each $e \in \mathcal{E}$, there is a corresponding point $k_e \in K$ with $e(k_e) \geq 0$. Then there is a point $k_0 \in K$ for which $e(k_0) \geq 0$ for all $e \in \mathcal{E}$. A simple proof of this result may be found in [15], Lemma 5.2.

Following [11], we claim that there exist states p_0 and q_0 on M_n such that

$$|\psi(\alpha v\beta)| \leq p_0(\alpha\alpha^*)^{1/2}\gamma_r(v)q_0(\beta^*\beta)^{1/2} \tag{2.8}$$

for all $\alpha \in M_{n,r}, \beta \in M_{r,n}$, and $v \in M_r(V)$ for which $\gamma_r(v) < \infty$, with $r \in \mathbb{N}$ arbitrary. If $\gamma_r(v) = 0$, then

$$\gamma_n(\alpha v\beta) \leq \|\alpha\|\,\gamma_r(v)\,\|\beta\| = 0,$$

i.e., $\lambda(\alpha v\beta) \in D_n$ for all scalars $\lambda > 0$. Since we chose ψ with $|\psi|_{D_n}| \leq 1$, we have $\lambda|\psi(\alpha v\beta)| \leq 1$ for all scalars $\lambda > 0$, hence $\psi(\alpha v\beta) = 0$ and (2.8) is trivial. Dividing by a constant, we may assume that $\gamma_r(v) = 1$. Summarizing, our task is to find states p_0 and q_0 on M_n such that for all $v \in M_r(V)$ with $\gamma_r(v) = 1$, we have that

$$|\psi(\alpha v\beta)| \leq p_0(\alpha\alpha^*)^{1/2}q_0(\beta^*\beta)^{1/2}. \tag{2.9}$$

It suffices to find states p_0, q_0 such that

$$\operatorname{Re}\psi(\alpha v\beta) \leq p_0(\alpha\alpha^*)^{1/2}q_0(\beta^*\beta)^{1/2}$$

since we can then replace α by $e^{i\theta}\alpha$ for a suitable $\theta \in [0, 2\pi]$. In turn, it is enough to prove that

$$\operatorname{Re}\psi(\alpha v\beta) \leq (1/2)[p_0(\alpha\alpha^*)+q_0(\beta^*\beta)]. \tag{2.10}$$

To see this, we replace α by $t^{1/2}\alpha$ and β by $t^{-1/2}\beta$ for $t > 0$. It follows that

$$\operatorname{Re}\psi(\alpha v\beta) \leq (1/2)[tp_0(\alpha\alpha^*) + t^{-1}q_0(\beta^*\beta)],$$

and minimizing for $t > 0$ (or simply letting $t = p_0(\alpha\alpha^*)^{-1/2}q_0(\beta^*\beta)^{1/2}$ for $\alpha \neq 0$), we obtain (2.9).

Letting S_n be the set of states on M_n, $S = S_n \times S_n$ is a compact and convex subset of $(M_n \oplus M_n)^*$. We write $A(S)$ for the continuous affine functions on S. Given $\alpha \in M_{n,r}$, $\beta \in M_{r,n}$ and $v \in M_r(V)$ with $\gamma_r(v) = 1$, we may define a corresponding function $e_{\alpha,v,\beta} \in A(K)$ by

$$e_{\alpha,v,\beta}(p,q) = p(\alpha\alpha^*) + q(\beta^*\beta) - 2\,\mathrm{Re}\,\psi(\alpha v\beta).$$

Letting $\mathcal{E}$ denote the collection of all such functions, we must show that there is a point $(p_0, q_0) \in S$ for which $e(p_0, q_0) \geq 0$ for all $e \in \mathcal{E}$. But we have that

i. Each function $e \in \mathcal{E}$ is non-negative at some point $(p_e, q_e) \in S$. To see this suppose that $e = e_{\alpha,v,\beta}$, and select states p_e and q_e with $p_e(\alpha\alpha^*) = \|\alpha\alpha^*\| = \|\alpha\|^2$, and $q_e(\beta^*\beta) = \|\beta\|^2$. Then we have

$$e_{\alpha,v,\beta}(p_e, q_e) = \|\alpha\|^2 + \|\beta\|^2 - 2\,\mathrm{Re}\,\psi(\alpha v\beta) \geq 0$$

since

$$\mathrm{Re}\,\psi(\alpha v\beta) \leq |\psi(\alpha v\beta)| \leq \|\alpha v\beta\| \leq \|\alpha\|\|\beta\| \leq (1/2)[\|\alpha\|^2 + \|\beta\|^2].$$

ii. The collection $\mathcal{E}$ is a cone of functions, i.e., $\mathcal{E} + \mathcal{E} \subseteq \mathcal{E}$, and $\lambda\mathcal{E} \subseteq \mathcal{E}$ for $\lambda \geq 0$. The second assertion is trivial. For the first we note that

$$e_{\alpha,v,\beta} + e_{\alpha',v',\beta'} = e_{\alpha'',v'',\beta''}$$

where $\alpha'' = \begin{bmatrix} \alpha & \alpha' \end{bmatrix}$, $\beta'' = \begin{bmatrix} \beta \\ \beta' \end{bmatrix}$, and $v'' = v \oplus v' \in M_{r+r'}$ satisfies $\gamma_{r+r'}(v \oplus v') = 1$.

From the convexity result, there exists a point (p_0, q_0) at which all of the functions in $\mathcal{E}$ are positive. Thus we have proved (2.8). We claim that we may perturb these states so that they are faithful. We recall that the normalized trace τ on M_n is faithful, i.e., we have that $\tau(\alpha^*\alpha) = 0$ implies that $\alpha = 0$. Given $0 < \varepsilon < 1$, it follows that $p = (1-\varepsilon)p_0 + \varepsilon\tau$ and $q = (1-\varepsilon)q_0 + \varepsilon\tau$ are faithful. Letting $\tilde{\psi} = (1-\varepsilon)\psi$, we have from (2.8) that if $\gamma_r(v) < \infty$,

$$\begin{aligned} |\tilde{\psi}(\alpha v\beta)| &\leq (1-\varepsilon)p_0(\alpha\alpha^*)^{1/2}q_0(\beta^*\beta)^{1/2}\gamma_r(v) \\ &\leq \frac{(1-\varepsilon)}{2}[p_0(\alpha\alpha^*) + q_0(\beta^*\beta)]\gamma_r(v) \\ &\leq \frac{1}{2}[p(\alpha\alpha^*) + q(\beta^*\beta)]\gamma_r(v). \end{aligned}$$

Replacing α by $t^{1/2}\alpha$ and β by $t^{-1/2}\alpha$, where t is a positive scalar, and then minimizing, we conclude that

$$|\tilde{\psi}(\alpha v\beta)| \leq p(\alpha\alpha^*)^{1/2}\gamma_r(v)q(\beta^*\beta)^{1/2}. \tag{2.11}$$

On the other hand, if we let ε be sufficiently small, we have from (2.7) that

$$\left|\tilde{\psi}|_{D_n}\right| \leq 1 < \left|\tilde{\psi}(v_0)\right|.$$

Applying the GNS theorem, we have corresponding faithful representations π and θ of M_n on finite dimensional Hilbert spaces H and K, respectively, with separating and cyclic vectors $\xi_0 \in H$ and $\eta_0 \in K$ satisfying $p(\alpha) = \langle\pi(\alpha)\xi_0|\xi_0\rangle$ and $q(\alpha) = \langle\theta(\alpha)\eta_0|\eta_0\rangle$, respectively, for all $\alpha \in M_n$.

Given a row matrix $\alpha = [\alpha_1, \alpha_2, \ldots, \alpha_n] \in M_{1,n}$, we define $\tilde{\alpha} \in M_n$ by

$$\tilde{\alpha} = \begin{bmatrix} \alpha_1 & \alpha_2 & \cdots \\ 0 & 0 & \cdots \\ \vdots & \vdots & \ddots \end{bmatrix}.$$

We let $\tilde{M}_{1,n}$ be the linear space of all such n by n matrices, and we let $H_0 = \pi(\tilde{M}_{1,n})\xi_0 \subseteq H$ and $K_0 = \theta(\tilde{M}_{1,n})\eta_0 \subseteq K$. Owing to the fact that ξ_0 and η_0 are separating, H_0 and K_0 are n-dimensional spaces.

Fixing an element $v \in V$, the sesquilinear form B_v defined on $K_0 \times H_0$ by

$$B_v(\theta(\tilde{\beta})\eta_0, \pi(\tilde{\alpha})\xi_0) = \tilde{\psi}(\alpha^* v\beta)$$

is well-defined since if, for example, $\theta(\tilde{\beta})\eta_0 = 0$, then since η_0 is separating and θ is a faithful representation, we have that $\tilde{\beta} = 0$ and $\tilde{\psi}(\alpha^* v\beta) = 0$. Thus there exists a unique linear map $\varphi(v) : K_0 \to H_0$ for which

$$\tilde{\psi}(\alpha^* v\beta) = \langle \varphi(v)\theta(\tilde{\beta})\eta_0 \mid \pi(\tilde{\alpha})\xi_0\rangle.$$

It is a simple matter to verify that the corresponding map $\varphi : V \to \mathcal{B}(K_0, H_0)$ is linear. Since H_0 and K_0 are n-dimensional, we may identify each of these spaces with $\mathbb{C}^n$, and φ with a mapping $\varphi : V \to M_n$.

Given a matrix $v \in M_n(V)$, we have that

$$v = \sum_{i,j} \tilde{e}^*_{1,i} v_{i,j} \tilde{e}_{1,j}$$

so

$$\begin{aligned} \tilde{\psi}(v) &= \sum \langle \varphi(v_{ij})\theta(\tilde{e}_{1,j})\eta_0 \mid \pi(\tilde{e}_{1,i})\xi_0\rangle \\ &= \langle \varphi_n(v)\eta \mid \xi\rangle, \end{aligned} \tag{2.12}$$

where

$$\eta = \begin{pmatrix} \theta(\tilde{e}_{1,1})\eta_0 \\ \vdots \\ \theta(\tilde{e}_{1,n})\eta_0 \end{pmatrix}, \ \xi = \begin{pmatrix} \pi(\tilde{e}_{1,1})\xi_0 \\ \vdots \\ \pi(\tilde{e}_{1,n})\xi_0 \end{pmatrix} \in \mathbb{C}^n$$

satisfy

$$\|\xi\|^2 = \sum \|\pi(\tilde{e}_{1,j})\xi_0\|^2 = \sum p(e^*_{1,j} e_{1,j}) = p(\varepsilon^{(n)}) = 1,$$

and similarly, $\|\eta\|^2 = 1$.

To show that φ satisfies (2.5), we must prove that if $v \in M_r(V)$ and $\gamma_r(v) < \infty$, then

$$|\langle \varphi_r(v)\eta_1 \mid \xi_1\rangle| \le \gamma_r(v)\|\eta_1\|\|\xi_1\|$$

for unit vectors ξ_1 and η_1 in $(\mathbb{C}^n)^r$. Letting

$$\xi_1 = \begin{pmatrix} \pi(\tilde{\alpha}_1)\xi_0 \\ \vdots \\ \pi(\tilde{\alpha}_r)\xi_0 \end{pmatrix}, \ \eta_1 = \begin{pmatrix} \theta(\tilde{\beta}_1)\eta_0 \\ \vdots \\ \theta(\tilde{\beta}_r)\eta_0 \end{pmatrix}$$

where α_i, $\beta_j \in M_{1,n}$, we have that

$$\|\xi_1\|^2 = \sum \|\pi(\tilde{\alpha}_i)\xi_0\|^2 = \sum p(\alpha_i^*\alpha_i) = p(\alpha^*\alpha),$$

and similarly $\|\eta_1\|^2 = q(\beta^*\beta)$, where

$$\alpha = \begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_r \end{bmatrix}, \ \beta = \begin{bmatrix} \beta_1 \\ \vdots \\ \beta_r \end{bmatrix} \in M_{r,n}.$$

It follows that

$$\begin{aligned}\langle \varphi_r(v)\eta_1 \mid \xi_1\rangle &= \sum \langle \varphi_0(v_{i,j})\theta(\tilde{\beta}_j)\eta_0 \mid \pi(\tilde{\alpha}_i)\xi_0\rangle \\ &= \sum \tilde{\psi}(\alpha_i^* v_{ij}\beta_j) \\ &= \tilde{\psi}(\alpha^* v\beta),\end{aligned}$$

and thus if $v \in D_r$,

$$\begin{aligned}|\langle \varphi_r(v)\eta_1 \mid \xi_1\rangle| &\leq |\tilde{\psi}(\alpha^* v\beta)| \\ &\leq p(\alpha\alpha^*)^{1/2}\gamma_r(v)q(\beta^*\beta)^{1/2} \\ &= \gamma_r(v)\|\xi_1\|\|\eta_1\| \\ &\leq 1.\end{aligned}$$

On the other hand we have that $v_0 \in M_n(V)$ satisfies

$$1 < \tilde{\psi}(v_0) = \langle \varphi_n(v_0)\eta \mid \xi\rangle,$$

and since that η and ξ are unit vectors, we have proved (2.6). □

Now, we know that gauges and absolutely convex sets are in 1-1 correspondence, so given a gauge γ we define the *dual gauge* γ° to be the Minkowski gauge of the polar of the unit set of γ, that is,

$$\gamma^\circ(w) = \sup\{|\langle v, w\rangle| : \gamma(v) \leq 1\}$$

for all w in W. Of course, since matrix gauges and absolutely matrix convex sets are also in 1-1 correspondence, we define the *dual matrix gauge* of a matrix gauge $\boldsymbol{\gamma}$ to be the Minkowski matrix gauge of the matrix polar of the matrix unit ball of $\boldsymbol{\gamma}$. That is to say

$$\boldsymbol{\gamma}^{\circledcirc}(w) = \sup\{\|\langle\langle v, w\rangle\rangle\| : \boldsymbol{\gamma}(v) \leq 1, v \in M_m(V), m \in \mathbb{N}\}$$

for all $w \in M_n(W)$.

Now let V be a locally convex space, and W its continuous dual $C(V, \mathbb{C})$. Then V and W are a dual pair of vector spaces. Let K be an absolutely convex subset of V, and then we can define (somewhat trivially) a matrix set $\boldsymbol{K}$ by letting $K_1 = K$, and $K_n = \varnothing$ for $n > 1$. Although $\boldsymbol{K}$ is clearly not absolutely matrix convex, we can build absolutely matrix convex sets from K.

We define the *minimal envelope* of K to be the double matrix polar of $\boldsymbol{K}$, i.e.

$$\hat{\boldsymbol{K}} := (\boldsymbol{K})^{\circledcirc\circledcirc}.$$

We define the *maximal envelope* of K to be the matrix polar of the polar of K,

$$\check{\boldsymbol{K}} := (\boldsymbol{K}^\circ)^{\circledcirc}.$$

In calling these sets minimal and maximal, we should check that they in fact are.

Proposition 2.2.2
If $\boldsymbol{L}$ is a weakly closed, absolutely matrix convex set with $L_1 = K$, then

$$\hat{\boldsymbol{K}} \subseteq \boldsymbol{L} \subseteq \check{\boldsymbol{K}}.$$

Proof :
We observe immediately, that $\boldsymbol{K} \subseteq \boldsymbol{L}$, so that $\boldsymbol{K}^{\circledcirc} \supseteq \boldsymbol{L}^{\circledcirc}$ and hence

$$\hat{\boldsymbol{K}} = \boldsymbol{K}^{\circledcirc\circledcirc} \subseteq \boldsymbol{L}^{\circledcirc\circledcirc} = \boldsymbol{L}$$

by the matrix bipolar theorem (Proposition 2.2.1).

In the other direction, we know that by 2.2 that $\boldsymbol{L}_1^{\circledcirc} = L_1^{\circ} = K^{\circ}$, so $\boldsymbol{L}^{\circledcirc} \supseteq \boldsymbol{K}^{\circ}$ and so

$$\boldsymbol{L} = \boldsymbol{L}^{\circledcirc\circledcirc} \subseteq \boldsymbol{K}^{\circ\circledcirc} = \check{\boldsymbol{K}}.$$

□

If ρ is the gauge in V with weakly closed unit set K, then we let $\hat{\boldsymbol{\rho}}$ and $\check{\boldsymbol{\rho}}$ be the matrix Minkowski gauges of $\hat{\boldsymbol{K}}$ and $\check{\boldsymbol{K}}$ respectively. We call these gauges the *maximal* and *minimal matrix gauges* of ρ, respectively, since by Proposition 2.2.2 we have that given any matrix gauge $\boldsymbol{\rho}$ with $\rho_1 = \rho$

$$\check{\boldsymbol{\rho}} \leq \boldsymbol{\rho} \leq \hat{\boldsymbol{\rho}}. \tag{2.13}$$

Note that the minimal envelope gives the maximal gauge and vice versa.

Example 2.2.2
If V is a normed space, with norm $\rho(v) = \|v\|$, the minimal and maximal matrix norms on V are defined by

$$\|v\|_{\min} = \sup\{\|\langle\langle v, w\rangle\rangle\| : w \in V^*, \|f\| \leq 1\} = \check{\boldsymbol{\rho}}(v)$$

and

$$\|v\|_{\max} = \sup\{\|\langle\langle v, w\rangle\rangle\| : w \in \mathcal{B}(V, M_p), \|w\| \leq 1, p \in \mathbb{N}\}$$

respectively. We call the operator spaces with these matrix norms $\min V$ and $\max V$ respectively. These agree with the standard definitions of min and max as given in [3] and [8].

2.3 The Hahn-Banach Theorem

We now turn our attention to another standard result of the classical theory of locally convex spaces, namely the Hahn-Banach extension theorem. It is worthwhile noting here that there is at present no known version of the Hahn-Banach separation theorem in the matrix convex setting—or indeed in the special cases of operator spaces or operator systems.

We recall that given a seminorm ρ on a vector space W, and a linear functional φ defined on a subspace V of W with

$$|\varphi(v)| \leq \rho(v)$$

for all $v \in V$, then the Hahn-Banach extension theorem says that there exists an extension of φ to a linear functional $\bar{\varphi}$ on W such that

$$|\bar{\varphi}(w)| \leq \rho(w)$$

for all $w \in W$.

We will prove a version of the Hahn-Banach theorem for (absolute) matrix gauges. Given a vector space W with a matrix gauge $\boldsymbol{\rho}$, we say that a subspace V of W is *cofinal* if given any $w \in W$ there are elements v_+ and v_- of V for which $\rho_1(v_+ + w)$ and $\rho_1(v_- - w)$ are finite.

Theorem 2.3.1
Let $\boldsymbol{\rho}$ be a matrix gauge on a vector space W, and φ a linear map defined from a cofinal subspace V of W to $\mathcal{B}(H)$ for some Hilbert space H, such that for any $n \in \mathbb{N}$,

$$\|\varphi_n(v)\| \leq \boldsymbol{\rho}(v)$$

for all $v \in M_n(V)$, then there exists an extension of φ to a linear map $\bar{\varphi}$ from W to $\mathcal{B}(H)$, such that for any $n \in \mathbb{N}$,

$$\|\bar{\varphi}_n(w)\| \leq \boldsymbol{\rho}(w)$$

for all $w \in M_n(W)$.

This will immediately give the obvious generalization of the classical result:

Corollary 2.3.2
Let $\boldsymbol{\rho}$ be a matrix seminorm on a vector space W, and φ a linear map defined from a subspace V of W to $\mathcal{B}(H)$ for some Hilbert space H, such that for any $n \in \mathbb{N}$,

$$\|\varphi_n(v)\| \leq \boldsymbol{\rho}(v)$$

for all $v \in M_n(V)$, then there exists an extension of φ to a linear map $\bar{\varphi}$ from W to $\mathcal{B}(H)$, such that for any $n \in \mathbb{N}$,

$$\|\bar{\varphi}_n(w)\| \leq \boldsymbol{\rho}(w)$$

for all $w \in M_n(W)$.

since $\boldsymbol{\rho}$ is finite, and hence any subspace V of W is cofinal.

The simplest way to prove Theorem 2.3.1 result is to rely on the Hahn-Banach theorem for (not necessarily absolute) matrix gauges of Effros and Winkler [15]. Given an $x \in \mathcal{B}(H)$ for some Hilbert space H, we say

$$\operatorname{Re} x \leq \lambda I$$

for $\lambda \in [0, \infty]$, if

$$\operatorname{Re}\langle x\xi|\xi\rangle \leq \lambda\langle\xi|\xi\rangle$$

for all $\xi \in H$.

Theorem 2.3.3
Let W be a vector space with a matrix gauge $\boldsymbol{\rho}$, and let H be a Hilbert space. Given a cofinal subspace $V \subset W$ and a linear mapping $\varphi : V \to \mathcal{B}(H)$ such that

$$Re\, \varphi_r(v) \leq \rho_r(v) I,$$

for all $v \in M_n(V)$ and $r \in \mathbb{N}$, then φ has a linear extension $\bar{\varphi} : W \to \mathcal{B}(H)$ such that

$$Re\, \bar{\varphi}(w) \leq \rho_r(w) I,$$

for all $w \in W$ and $r \in \mathbb{N}$.

We are now in a position to prove Theorem 2.3.1.
Proof (Theorem 2.3.1):
We consider the auxiliary map $\psi : V \to M_2(\mathcal{B}(H))$ given by

$$v \mapsto \begin{bmatrix} 0 & \varphi(v) \\ 0 & 0 \end{bmatrix},$$

following Wittstock. We note that

$$\operatorname{Re} \psi_n(v) \leq \frac{1}{2}\rho_n(v)$$

and clearly any subspace which is cofinal for $\boldsymbol{\rho}$ is cofinal for the matrix gauge $\frac{1}{2}\boldsymbol{\rho}$. So Theorem 2.3.3 tells us that there is an extension $\bar{\psi} : W \to M_2(\mathcal{B}(H))$ of ψ, such that

$$\operatorname{Re} \psi_n(w) \leq \frac{1}{2}\rho_n(w)$$

for all $w \in W$. We note that

$$\bar{\varphi} = \begin{bmatrix} 1 & 0 \end{bmatrix} \varphi \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

is an extension of φ to all of W, and

$$\|\bar{\varphi}_n(v)\|_n \leq 2\|\bar{\psi}_n(v)\| \leq 2 \sup_{\|\xi\|\leq 1} |\langle \bar{\psi}_n(v)\xi|\xi\rangle| \leq 2\frac{1}{2}\rho_n(v)\|I\|,$$

whence the result. □

We note that this proof uses the now standard 2 by 2 matrix trick pioneered by Wittstock. It should be possible to provide a direct proof using the matrix bipolar theorem (Proposition 2.2.1), but the author is of the opinion that this would not be particularly enlightening.

Example 2.3.1
Corollary 2.3.2 tells us that if V is a subspace of an operator space W, and H is a Hilbert space, then for any complete contraction $\varphi : V \to \mathcal{B}(H)$, we have a completely contractive extension $\bar{\varphi} : W \to \mathcal{B}(H)$. So the Arveson-Wittstock Hahn-Banach theorem [1] is a special case of this result.

2.4 M_∞-Convexity

At this point, we introduce an alternative way of looking a matrix convexity. This viewpoint is very much in the spirit of unpublished work of Johnson [20] on C*-operator spaces, which we will introduce in Chapter 4 to discuss compactness results. The key idea is that all the important notions we have discussed so far in terms of a vector space V and the matrix spaces $M_n(V)$ over it have corresponding versions for the M_∞-bimodule $M_\infty(V)$ (recall from Section 1.2 that M_∞ is the space of infinite matrices with only finitely many non-zero entries).

If V is vector space, and $\boldsymbol{\gamma}$ is a matrix gauge, then there is a corresponding gauge γ_∞ defined on $M_\infty(V)$ by letting

$$\gamma_\infty(v) = \gamma_n(v)$$

where v is zero outside the first n by n entries. This is well-defined by Axiom (AMG1). It is not surprising that the matrix gauges on $M_\infty(V)$ which come from matrix gauges in this fashion are special. To characterize these, we introduce the idea of an orthogonal set of elements of $M_\infty(V)$: $v_1, \ldots, v_n \in M_\infty(V)$ are *orthogonal* if there exist orthogonal projections $e_1, \ldots, e_n \in M_\infty$ such that $e_i v_i e_i = v_i$.

Lemma 2.4.1
If $\boldsymbol{\gamma}$ is a matrix gauge, then the corresponding gauge γ_∞ satisfies

(AM_∞G1) $\quad \gamma_\infty(v + w) = \max\{\gamma_\infty(v), \gamma_\infty(w)\}\quad$ *for all orthogonal v, $w \in M_\infty(V)$*

(AM_∞G2) $\quad \gamma_\infty(\alpha^* v\beta) \leq \|\alpha\|\gamma_\infty(v)\|\beta\|\quad$ *for all $v \in M_\infty(V)$, α, $\beta \in M_\infty$.*

Proof :

The only difficulty is in showing (AM_∞G1), since (AM_∞G2) follows immediately from (AMG2). If v and w are orthogonal, with orthogonal projections p and q such that $pvp = v$ and $qwq = w$, then we have that there must be some $m \in \mathbb{N}$ so that v, $w \in M_m(V)$ and p, $q \in M_m$. We therefore know that there are partial isometries α and β which implement equivalences of the projections p and p_k and q and p_l respectively. So we must have $\alpha v\alpha^* \in M_k(V)$ and $\beta w\beta^* \in M_l(V)$, so

$$v + w = \begin{bmatrix}\alpha^* & \beta^*\end{bmatrix}\begin{bmatrix}\alpha v\alpha^* & 0\\ 0 & \beta v\beta^*\end{bmatrix}\begin{bmatrix}\alpha\\ \beta\end{bmatrix}.$$

This tells us that

$$\gamma_\infty(v + w) = \gamma_\infty\left(\begin{bmatrix}\alpha v\alpha^* & 0\\ 0 & \beta v\beta^*\end{bmatrix}\right) \leq \gamma_{k+l}(v \oplus w) = \max\{\gamma_\infty(v), \gamma_\infty(w)\}.$$

□

We will call any gauge on $M_\infty(V)$ which satisfies (AM$_\infty$G1) and (AM$_\infty$G2) an M_∞-*module gauge*. Given such a gauge γ it is immediate that

$$\gamma_n = \gamma|_{M_n(V)}$$

determines a matrix gauge on V, and that the correspondence is one to one.

Definition 2.4.1
*A set $K \subseteq M_\infty(V)$ is said to be M_∞-*convex *if it satisfies the conditions*

(AM$_\infty$C1)	$v + w \in K$	*for all orthogonal v, $w \in K$*
(AM$_\infty$C2)	$\alpha K \beta \subseteq K$	*for any contractions α, $\beta \in M_\infty$.*

It is clear that M_∞-convex sets are the unit sets of M_∞-module gauges, and that the Minkowski gauges of M_∞-convex sets are M_∞-module gauges. This means that M_∞-convex sets in $M_\infty(V)$ are in a one-to-one correspondence with matrix convex sets in V. This correspondence can be given concretely via

$$K \mapsto \boldsymbol{K} = (K_n = p_n K p_n) \tag{2.14}$$

and

$$\boldsymbol{K} \mapsto K = \bigcup_{n=1}^{\infty} K_n. \tag{2.15}$$

We note that there is no correspondence between arbitrary matrix sets in V and sets in $M_\infty(V)$.

We can re-cast the matrix bipolar theorem (Proposition 2.2.1) in terms of M_∞-convexity. Let V and W be two locally convex spaces in duality. Given a subset S of $M_\infty(V)$ we define it's M_∞-*polar* $S^{\circledcirc} \subseteq M_\infty(W)$ to be the set

$$S^{\circledcirc} = \{w \in M_\infty(W) : \|\langle\langle v, w\rangle\rangle\| \leq 1 \text{ for all } v \in S\}.$$

$S^{\circledcirc}$ is M_∞-convex, since if w, $w' \in S^{\circledcirc}$ are orthogonal, with orthogonal projections p and $q \in M_\infty$ so that $pwp = w$ and $qw'q = w'$, then given $v \in M_n(V) \cap S$, we have $\tilde{p} = p_n \otimes p$ and $\tilde{q} = p_n \otimes q$ are orthogonal projections in $M_\infty \otimes M_\infty$ such that $\tilde{p}\langle\langle v, w\rangle\rangle\tilde{p} = \langle\langle v, w\rangle\rangle$, $\tilde{q}\langle\langle v, w'\rangle\rangle\tilde{q} = \langle\langle v, w'\rangle\rangle$ and

$$\begin{aligned}\|\langle\langle v, pwp + qw'q\rangle\rangle\| &= \|\tilde{p}\langle\langle v, w\rangle\rangle\tilde{p} + \tilde{q}\langle\langle v, w'\rangle\rangle\tilde{q}\| \\ &= \max\{\|\langle\langle v, w\rangle\rangle\|, \|\langle\langle v, w'\rangle\rangle\|\} \\ &\leq 1.\end{aligned}$$

So $S^{\circledcirc}$ satisfies (AM$_\infty$C1). On the other hand, given two contractions α, $\beta \in M_\infty$, then so are $\tilde{\alpha} = p_n \otimes \alpha$ and $\tilde{\beta} = p_n \otimes \beta$, and so

$$\|\langle\langle v, \alpha w \beta\rangle\rangle\| = \|\tilde{\alpha}\langle\langle v, w\rangle\rangle\tilde{\beta}\| \leq \|\langle\langle v, w\rangle\rangle\| \leq 1.$$

So $S^{\circledcirc}$ satisfies (AM$_\infty$C2), and we are done. We now have what we need to prove an M_∞-module version of the bipolar theorem.

Proposition 2.4.2
Given a subset S of $M_\infty(V)$, $S^{\circledcirc\circledcirc}$ is the smallest weakly closed M_∞-convex set containing S.

Proof :

Let K be the smallest M_∞-convex set containing S, and let $\bar{K}$ be its weak closure, so we want to show that $K^{\circledcirc\circledcirc} = \bar{K}$. Let $\boldsymbol{K}$ be the matrix convex set corresponding to K via (2.14). We note that $\bar{K}$ corresponds to the weak closure of $\boldsymbol{K}$ via (2.15), since if v_λ is a net in K converging weakly to $v \in M_\infty(V)$, then if $v \in M_n(V)$,

$$v = \lim_\lambda p_n v_\lambda p_n \in \bar{K}_n.$$

So if there is a $v_0 \in B^{\circledcirc\circledcirc} \setminus \bar{B}$, then there is an n with $v_0 \in M_n(V)$, and so $v_0 \notin \bar{B}_n$. Now from the matrix bipolar theorem, there must be a $w_0 \in M_n(W)$ so that

$$||\langle\langle v_0, w_0\rangle\rangle|| > 1 \geq ||\langle\langle v, w_0\rangle\rangle||$$

for all $v \in B_r$, with r arbitrary, that is to say, $v \in B$. So if we think of w_0 as being an element of $M_\infty(W)$, we have $w_0 \in B^{\circledcirc}$, and so $v_0 \notin B^{\circledcirc\circledcirc}$. □

Similarly, the correspondence between matrix gauges and M_∞-module gauges allows us to cheaply prove a Hahn-Banach separation theorem for this situation.

Proposition 2.4.3
Let ρ be a M_∞-module gauge on $M_\infty(W)$, and φ a linear map defined from a cofinal subspace V of W to $\mathbb{C}$, such that,

$$||\varphi_\infty(v)|| \leq \rho(v)$$

for all $v \in M_\infty(V)$, then there exists an extension of φ to a linear map $\bar{\varphi}$ from W to $\mathbb{C}$, such that for any $n \in \mathbb{N}$,

$$||\bar{\varphi}_\infty(w)|| \leq \rho(w)$$

for all $w \in M_\infty(W)$.

Proof :
Letting $\boldsymbol{\rho}$ be the matrix gauge corresponding to ρ, we get that φ is a linear map from V to $\mathbb{C}$ satisfying

$$||\varphi_n(v)|| \leq \rho_n(v)$$

so it can be extended to all of W so that

$$||\bar{\varphi}_n(w)|| \leq \rho_n(w)$$

and the equivalence of the gauges then gives us

$$||\bar{\varphi}_\infty(w)|| \leq \rho(w)$$

as required. □

Chapter 3

Local Operator Spaces

All the work of the previous chapter has been leading up to us being able to define and work with spaces which bear the same relationship to locally convex spaces that operator spaces do to Banach spaces. In this chapter we define them and develop the theory, giving analogues of most of the fundamentals of locally convex space theory, and proving analogues of the Arens-Mackey duality theorem, the uniform boundedness principle and the Schwartz kernel theorem. We also investigate the class of spaces corresponding to nuclear spaces and show that they are extremely well-behaved.

We note that the two most important techniques used in this chapter are that any local operator space is a projective limit of operator spaces, which often allows us to lift techniques, definitions and results up from the theory of operator spaces, and the fact that local operator spaces can be identified with a special class of topological M_∞-bimodules, where local convexity is replaced by the M_∞-convexity. This second approach allows the use of some results from the classical theory in working with local operator spaces.

We hope that local operator spaces will provide an axiomatic environment which we hope will be of use in a number of areas of noncommutative mathematics, and we indicate their potential via a number of provocative examples throughout this chapter.

3.1 Local Operator Spaces

Let V be a locally convex topological vector space, and let $\mathfrak{R}$ be a collection of matrix seminorms such that ρ_n is a continuous seminorm (when $M_n(V)$ is identified with V^{n^2}) for all $\boldsymbol{\rho} \in \mathfrak{R}$, and the family of seminorms $\{\rho_1 : \boldsymbol{\rho} \in \mathfrak{R}\}$ generate the topology of V, then we say the $\mathfrak{R}$ is a *local operator structure* on V. Given such a local operator structure on V, we say that a matrix seminorm $\boldsymbol{\gamma}$ on V is *matrix continuous* if there exist matrix seminorms $\boldsymbol{\rho^1}, \ldots, \boldsymbol{\rho^p}$ and some constant $k > 0$ such that

$$\gamma_n \leq k(\boldsymbol{\rho^1} + \cdots + \boldsymbol{\rho^p})_n$$

for all $n \in \mathbb{N}$ (note that k is independent of n). We denote the set of all such matrix seminorms by $\mathfrak{N}_\mathfrak{R}$, and note that this set is partially ordered by the relation $\boldsymbol{\gamma} \leq \boldsymbol{\rho}$ when

$$\gamma_n(v) \leq \rho_n(v)$$

for all $v \in M_n(V)$ and $n \in \mathbb{N}$.

We have an equivalence relation on families of matrix seminorms on a vector space V: two such families $\mathfrak{R}$, $\mathfrak{R}'$ are equivalent if the families of seminorms $\{\rho_1 : \boldsymbol{\rho} \in \mathfrak{R}\}$ and $\{\rho_1 : \boldsymbol{\rho} \in \mathfrak{R}'\}$ determine the same topology on V and if every matrix seminorm in $\mathfrak{R}$ is matrix continuous with respect to $\mathfrak{R}'$ and vice versa (or, equivalently, $\mathfrak{N}_\mathfrak{R} = \mathfrak{N}_{\mathfrak{R}'}$). We call an equivalence class under this equivalence relation a *matrix topology* on V. An element of the matrix topology is called a *generating family* of matrix seminorms. We say that a matrix topology is *countably generated* if it has a countable generating family.

Definition 3.1.1
A local operator space *is a vector space V together with a matrix topology.*

Given a locally convex space V, a quantization *of V is a local operator space whose vector space is V, and whose matrix topology is generated by some local operator structure $\mathfrak{R}$.*

We denote the family of all matrix continuous seminorms of a local operator space V by $\mathfrak{N}(V)$. In practice, however, we will usually work with a convenient generating family, just as in the classical theory. The first level of an operator space is simply a locally convex space, and the local operator space is a quantization of its first level.

Definition 3.1.2
Given two local operator spaces V and W, we say that a linear mapping $\varphi : V \to W$ is matrix continuous[1] *if for every matrix seminorm $\sigma \in \mathfrak{N}(W)$, there exists a matrix seminorm $\rho \in \mathfrak{N}(V)$ such that*

$$\sigma_n(\varphi_n(v)) \leq \rho_n(v)$$

for all $v \in M_n(V)$ and $n \in \mathbb{N}$. If, in addition, φ is a bijection with matrix continuous inverse, we say that it is a matrix homeomorphism.

We denote the space of all matrix continuous maps from V to W by $\mathcal{CC}(V, W)$.

To show a map is matrix continuous it suffices to check for matrix seminorms in generating families. More precisely, given generating families $\mathfrak{R} \in \mathfrak{N}(V)$ and $\mathfrak{R}' \in \mathfrak{N}(W)$, a map φ is matrix continuous if and only if for all $\boldsymbol{\sigma} \in \mathfrak{R}'$, there is are $\boldsymbol{\rho^1}, \dots, \boldsymbol{\rho^q} \in \mathfrak{R}$ and a $k > 0$, such that

$$\sigma_n(\varphi_n(v)) \leq k(\rho^1_n(v) + \cdots + \rho^q_n(v))$$

for all $v \in M_n(V)$ and $n \in \mathbb{N}$.

Example 3.1.1
If V is an operator space, then it is a local operator space with the matrix topology generated by its matrix norm $\|\cdot\|$. The matrix continuous maps between two operator spaces are then simply the completely bounded maps, and matrix homeomorphisms are complete isomorphisms.

If V is a Banach space, then $\mathfrak{N}(\max V)$ contains all the matrix norms for quantizations of V.

There is no collection of maps corresponding to completely contractive maps, since the matrix topology forgets the metric structure associated with the original norm, as one would expect.

A locally convex space may have many different (i.e. non-matrix homeomorphic) quantizations. As a simple example consider a Hilbert space H with the locally convex topology determined by its norm: we know that it if H is infinite dimensional that H_c and H_r are non-completely isomorphic operator spaces, and as Example 3.1.1 shows these are quantizations of H which are not matrix homeomorphic.

Given a locally convex space V we have the extremal quantizations $\min V$ and $\max V$ defined as follows. If ρ is any continuous seminorm on V, then we know that its unit ball is closed and convex, and so is closed in the weak topology $\sigma(V, V')$ on V determined by its continuous dual. So we may define the minimal and maximal matrix seminorms associated with ρ, namely $\hat{\boldsymbol{\rho}}$ and $\check{\boldsymbol{\rho}}$. We let $\min V$ be the local operator space determined by the matrix seminorms $\{\check{\boldsymbol{\rho}} : \rho \in \mathfrak{N}(V)\}$, and we let $\max V$ be the local operator space determined by the matrix seminorms $\{\hat{\boldsymbol{\rho}} : \rho \in \mathfrak{N}(V)\}$. It is immediate from (2.13) that the identity map on V induces the following diagram of matrix continuous mappings

$$\max V \to \boldsymbol{V} \to \min V \tag{3.1}$$

where $\boldsymbol{V}$ is *any* quantization of V.

[1] The terminology here is not optimal: we would desire to call such maps *completely continuous* so as to agree with the terms *completely bounded* and *completely positive*, however the term completely continuous is already in use in the literature on locally convex spaces.

If V is a subspace of a local operator space W, then it is clear that the restrictions of the operator seminorms in $\mathfrak{N}(W)$ determine a matrix topology on V. If V is closed in W at the first level (or $M_n(V)$ is closed in $M_n(W)$ for some n), then $M_m(V)$ is a closed subspace of $M_m(W)$ for any $m \in \mathbb{N}$.

If V and W are local operator spaces, then a matrix continuous map $\varphi \in \mathcal{CC}(V, W)$ which is a matrix homeomorphism onto its image in the subspace matrix topology, is called an *inclusion of the local operator space V into W*.

If V is a subspace of W, then for every $\boldsymbol{\rho} \in \mathfrak{N}(V)$, then we have a *quotient matrix seminorm* $\tilde{\boldsymbol{\rho}}$ given by the quotient seminorms $\tilde{\rho}_n$ of the ρ_n on

$$M_n(W/V) = M_n(W)/M_n(V).$$

So W/V gains a matrix topology determined by $\{\tilde{\boldsymbol{\rho}} : \boldsymbol{\rho} \in \mathfrak{N}(V)\}$. If V and W are local operator spaces, any matrix continuous map $\varphi \in \mathcal{CC}(V, W)$ which induces a matrix homeomorphism $\tilde{\varphi} : V/\ker\varphi \to W$ is called a *matrix quotient map* from V to W.

We say that V is an *operator Frechet space* if it is complete and the matrix topology is countably generated. In particular this implies that each $M_n(V)$ is a Frechet space. Since quotients of Frechet spaces by closed subspaces are Frechet—with generating family of seminorms being the quotient seminorms—we have that the quotient of an operator Frechet space by a closed subspace is also an operator Frechet space.

Example 3.1.2
If A_θ is a non-commutative torus as in Example 2.1.2, then we let A_θ^∞ be the vector space of elements of A_θ such that $D^{k,l}$ is defined for all k, $l \in \mathbb{N}$. Then all the matrix gauges $\boldsymbol{\rho^{k,l}}$ are finite, so they are matrix seminorms. The same follows for the gauges $\boldsymbol{\gamma^i}$. So we can turn A_θ^∞ into an operator space with the matrix topology generated by the $\boldsymbol{\rho^{k,l}}$. It is trivial that this matrix topology is the same as the matrix topology generated by the $\boldsymbol{\gamma^i}$. Note, however, that the $\boldsymbol{\gamma^i}$ norms have the additional nice property that they are partially ordered (indeed, they are totally ordered). The significance of this will become apparent in the next section.

The maps $D^{k,l}$ are now matrix continuous maps from A_θ^∞ to itself. They are matrix quotient maps, but not matrix homeomorphisms.

Example 3.1.3
The construction of the previous example trivially generalizes to the generality of Example 2.1.3, although the $\boldsymbol{\gamma^k}$ are merely partially ordered. We will denote such a space as V_σ^∞.

Moreover, we no longer need to restrict ourselves to G acting on operator spaces V. We may instead consider strongly continuous action σ of G on a local operator space V. Then we obtain matrix seminorms $\boldsymbol{\rho^\mu}$ on the smooth elements of V of the form

$$\rho_n^\mu(v) = \rho_n(D_n^\mu(v))$$

where $\boldsymbol{\rho} \in \mathfrak{N}(V)$, and $\boldsymbol{\gamma^{\rho,k}}$ where

$$\gamma_n^{\boldsymbol{\rho},k}(v) = \sup_{\substack{\boldsymbol{\rho}' \in \mathfrak{N}(V) \\ \boldsymbol{\rho}' \leq \boldsymbol{\rho}}} \sum_{\|\mu\| \leq k} \frac{\rho_n'(D_n^\mu(v))}{\mu!}$$

for $\boldsymbol{\rho} \in \mathfrak{N}(V)$. Again we will denote the space of smooth vectors V_σ^∞.

Example 3.1.4
Let $\mathcal{S}(\mathbb{N}^d)$ be the Schwartz space on $\mathbb{N}^d$. The topology on $\mathcal{S}(\mathbb{N}^d)$ is generated by a number of equivalent families of seminorms, in particular $\{\gamma^n : n \in \mathbb{N}\}$ where

$$\gamma^n(v) = \sup_{k \in \mathbb{N}^d} (d + |k|)^n |v(k)|$$

for $v \in \mathcal{S}(\mathbb{N}^d)$, and $\{\rho^n : n \in \mathbb{N}\}$ where

$$\rho^n(v) = \sum_{k \in \mathbb{N}^d} (d + |k|)^n |v(k)|^2.$$

But we may turn γ^n into a matrix seminorm by embedding $\mathcal{S}(\mathbb{N}^d)$ into $\mathcal{B}(\ell^2(\mathbb{N}^d))$ by letting $\{e_k : k \in \mathbb{N}^d\}$ be the standard orthonormal basis for $\ell^2(\mathbb{N}^d)$, and making v act by

$$v \cdot e_k = (d + |k|)^n v(k) e_k$$

ie. putting $(d + |k|)^n v(k)$ down the diagonal. Then the norm of v agrees with $\gamma^n(v)$, and so we define $\boldsymbol{\gamma^n}$ to be the matrix seminorm we get by pulling back the matrix norm on $\mathcal{B}(\ell^2(\mathbb{N}^d))$. Similarly, we may embed $\mathcal{S}(\mathbb{N}^d)$ into $\mathcal{B}(\ell^2(\mathbb{N}^d))$ by letting v act by

$$v \cdot e_k = (d + |k|)^{n/2} v(k) e_1$$

ie. by putting $(d + |k|)^{n/2} v(k)$ into the first row. Then the norm on $\mathcal{B}(\ell^2(\mathbb{N}^d))$ agrees with $\rho^n(v)$, and so we get another family of matrix seminorms $\boldsymbol{\rho^n}$.

We also note that if $d = 2$, then $A_\theta^\infty \cong \mathcal{S}(\mathbb{N}^d)$ and so we get a third potential quantization. We could also have embedded v into the first column, and achieved a fourth quantization.

We would expect that these quantizations of $\mathcal{S}(\mathbb{N}^d)$ would be quite distinct since we are quantizing the seminorms in distinctly different ways: roughly speaking, the $\boldsymbol{\gamma^n}$ have been quantized as C*-matrix norms, while the $\boldsymbol{\rho^n}$ have been quantized as Hilbert row matrix norms. If these were operator spaces, we would expect these quantizations to be incomparable. However, as we will see, these different quantizations give the same matrix topology.

Example 3.1.5
If H is a Hilbert space, and A is a family of densely defined closable operators on H with common domain $\mathcal{D}$ which contains the identity, then we call A an *O-family* (or an *Op-family*) following Schmüdgen [36]. We define the *matrix graph topology* on $\mathcal{D}$ to be the matrix topology generated by the matrix seminorms of Example 2.1.4 where $V = H_c$ (note that they are indeed matrix seminorms on this subspace) and denote this local operator space by $\mathcal{D}_{A,c}$. This is clearly a natural quantization of the *graph topology* $\mathcal{D}_A$ of [36].

Moreover, every element $a \in A$ is now seen to be an element of $\mathcal{CC}(\mathcal{D}_{A,c}, H)$ by looking at the definition of the seminorms, and equally clearly it is the weakest such.

Example 3.1.6
Consider a one particle quantum mechanical system with position and momentum operators p and q acting via the Schrödinger representation on $L^2(\mathbb{R}^3)$. p and q are closable unbounded operators with dense domain $\mathcal{S}(\mathbb{R}^3)$, and moreover the map $\mathcal{S}(\mathbb{R}^3)$ to $\mathcal{S}(\mathbb{R}^3)$, so that we can compose them, so we can consider the unital algebra $\mathcal{A}$ of unbounded operators generated by p and q. The graph topology $W_{\mathcal{A}}$ on W is the same as the standard topology of $\mathcal{S}(\mathbb{R}^3)$ (see [6] for details). Then given Example 3.1.5 we have a natural matrix topology on the *wave functions* W, namely $W_{\mathcal{A},c}$, and this is a quantization of $\mathcal{S}(\mathbb{R}^3)$.

For more general quantum mechanical systems, we have that W is a countably generated nuclear space (see Section 3.9) sitting densely inside some Hilbert space and is a common dense domain for the algebra of observables, so we can apply the above construction to topologize W. In this context, $\mathcal{A}$ is implicit, so we will often simply write W_c.

3.2 Projective Limits

When working with locally convex spaces, it is often convenient to consider them as projective limits of Banach spaces, since this allows us to use results from Banach space theory to derive results for locally convex spaces. This leads us to the conclusion that it would be useful to have such a theory available for local operator spaces, especially given that the theory of operator spaces is comparatively well developed.

Recall that, classically, if we have a vector space V and a family of locally convex spaces V_γ and linear maps $\pi_\gamma : V \to V_\gamma$, then we may associate to each $\gamma \in \Gamma$ and $\rho \in \mathfrak{N}(V_\gamma)$ the seminorm $\rho \circ \pi_\gamma$ on V. Then this family of seminorms generates the *projective locally convex topology* on V, which and we denote V with this topology as

$$V = \varprojlim V_\gamma = \varprojlim\{(V_\gamma, \pi_\gamma) : \gamma \in \Gamma\}.$$

Conversely if the topology on V is generated by a family of seminorms $\mathfrak{R}$, then we have

$$V = \varprojlim\{(\bar{V}_\rho, \pi_\rho) : \rho \in \mathfrak{R}\}$$

where $\bar{V}_\rho$ is the completion of the space V_ρ in (2.1). So any V is a projective limit of Banach spaces. Note that taking completions is not necessary, but is customary since is simplifies working with the spaces.

It is often most convenient to work with Γ a directed set, with V_γ Banach spaces so that there are connecting maps $\pi_{\gamma,\rho} : V_\rho \to V_\gamma$ for $\rho \geq \gamma$ such that

$$\pi_{\gamma,\rho} \circ \pi_\rho = \pi_\gamma$$

and which are contractions. We can certainly always do this, since $\mathfrak{N}(V)$ is directed by the obvious relationship

$$\gamma \leq \rho \iff \gamma(v) \leq \rho(v), \ \forall v \in V \tag{3.2}$$

but we can usually get away with a much smaller generating family. In fact if $\mathfrak{R} \subseteq \mathfrak{N}(V)$ is a generating family which is a directed set with the relationship $\leq$ of (3.2), then this is sufficient to guarantee that the connecting maps are contractions.

One advantage of working with directed sets is that if we have two locally convex spaces $V = \varprojlim(V_\rho, \pi_\rho)$ and $W = \varprojlim(V_\sigma, \theta_\sigma)$ where the projective limits are directed, then a linear map $\varphi : V \to W$ is continuous if and only if for each σ there is a ρ and a $k > 0$ such that

$$\|\theta_\sigma(\varphi(v))\| \leq k\|\pi_\rho(v)\|.$$

Similarly we can define the *projective local operator space topology* on a vector space V determined by a family of local operator spaces V_γ and linear mappings $\pi_\gamma : V \to V_\gamma$ by defining matrix seminorms $\boldsymbol{\rho} = \boldsymbol{\sigma} \circ \boldsymbol{\pi_\gamma}$ for $\gamma \in \Gamma$ and $\sigma \in \boldsymbol{\mathfrak{N}}(V)$, and we write

$$V = \mathop{\mathbf{\varprojlim}} V_\gamma = \mathop{\mathbf{\varprojlim}}(V_\gamma, \pi_\gamma).$$

Once again we can go in the converse direction, so that if V is an arbitrary local operator space, then

$$V = \mathop{\mathbf{\varprojlim}}\{(\bar{V}_{\boldsymbol{\rho}}, \pi_{\boldsymbol{\rho}}) : \boldsymbol{\rho} \in \boldsymbol{\mathfrak{N}}(V)\}$$

where $V_{\boldsymbol{\rho}}$ is the completion of matrix normed space $V_{\boldsymbol{\rho}}$ as in (2.3), and is hence an operator space.

We can, if it is useful, insist that the limit be taken over a directed set of operator spaces so that for $\rho \geq \gamma$ there are connecting maps $\pi_{\gamma,\rho} : V_\rho \to V_\gamma$ which are complete contractions, since this is the case if we order $\boldsymbol{\mathfrak{N}}(V)$ by saying

$$\boldsymbol{\rho} \leq \boldsymbol{\sigma} \iff \rho_n(v) \leq \sigma_n(v), \ \forall v \in M_n(V), \ n \in \mathbb{N}.$$

But again, any generating family for the matrix topology which is a directed family under $\leq$ will suffice.

As we might expect, we can use this to provide a nice characterization of matrix continuity. If $V = \mathop{\mathbf{\varprojlim}}(V_\rho, \pi_\rho)$ and $W = \mathop{\mathbf{\varprojlim}}(W_\sigma, \theta_\sigma)$ are directed, then $\varphi : V \to W$ is matrix continuous if and only if for each σ there is a ρ and a $k > 0$ such that

$$\|(\theta_\sigma)_n(\varphi_n(v))\|_n \leq k\|(\pi_\rho)_n(v)\|_n$$

for all $v \in M_n(V)$ and $n \in \mathbb{N}$.

We should note that in all the above we are taking projective limits of the topologies, not the underlying vector spaces. This is key in the following result.

Lemma 3.2.1
If V is a finite dimensional locally convex space, then all quantizations of V are identical.

Proof :
Choose any quantization of V, and write it as a projective limit of of operator spaces $V = \varprojlim V_\gamma$. Without loss of generality the V_γ are of the same dimension as V. But then the matrix topologies on the V_γ are all identical, since any two operator spaces of the same dimension are completely isomorphic, and so the limit matrix topology is the same as the matrix topologies on the V_γ (this observation is due to S. Winkler). □

If W is a subspace of a local operator space $V = \varprojlim(V_\rho, \pi_\rho)$ then we have that

$$W = \varprojlim(W_\rho, \tilde{\pi_\rho})$$

where $W_\rho = \pi_\rho(W)$ and $\tilde{\pi_\rho} = \pi_\rho|_W$.

Note that all of the above presuppose the existence of the original vector space V, and we may want to take a projective limit over an arbitrary projective system of local operator spaces. In this case we again use the distinction between the limit of the vector spaces and the limit of the topologies. If we are given V_γ where γ lie in a directed set Γ, and for $\rho > \gamma$ we have a matrix continuous connecting map $\pi_{\gamma,\rho} : V_\rho \to V_\gamma$ then we simply take the vector space projective limit $\varprojlim V_\gamma$ to be our V, and then continue with the original construction. We will need this sort of construction in later sections.

We now have a proposition which will be of importance in later sections.

Proposition 3.2.2
If V is a locally convex space, then

$$\min V = \varprojlim\{\min \bar{V_\rho} : \rho \in \mathfrak{N}(V)\}$$

and

$$\max V = \varprojlim\{\max \bar{V_\rho} : \rho \in \mathfrak{N}(V)\}.$$

Indeed, we can replace $\mathfrak{N}(V)$ with any generating family which is directed by $\leq$.

Proof :
It suffices that if $\rho \in \mathfrak{N}(V)$ then

$$\bar{V_{\hat{\rho}}} = \max \bar{V_\rho}$$

and

$$\bar{V_{\check{\rho}}} = \min \bar{V_\rho}$$

by the definition of $\min V$ and $\max V$. The second we observe follows since the adjoint of the quotient map π_ρ maps the unit ball of $\bar{V_\rho}^*$ onto the polar $(B^\rho)^\circ$ of the unit set of ρ, so that we have for any $v \in M_n(V)$

$$\begin{aligned}
\check{\rho}_n(v) &= \sup\{\|\langle\langle v, w\rangle\rangle\| : w \in (B^\rho)^\circ\} \\
&= \sup\{\|\langle\langle(\pi_\rho)_n(v), x\rangle\rangle\| : x \in \bar{V_\rho}^*,\ \|x\| \leq 1\} \\
&= \|(\pi_\rho)_n(v)\|_{\min}
\end{aligned}$$

But we also have that the quotient mapping π_ρ inflates to a quotient mapping $(\pi_\rho)_n : M_n(V) \to M_n(\bar{V_\rho}$ and so we have the adjoint at the nth level mapping the unit ball of $M_n(\bar{V_\rho}^*) = \mathcal{B}(\bar{V_\rho}, M_n)$ onto $(B^\rho)^{\circledcirc}_n$. Hence we can say that

$$\begin{aligned}
\hat{\rho}_m &= \sup\{\|\langle\langle v, w\rangle\rangle\| : w \in B^\rho)^{\circledcirc}_n,\ n \in \mathbb{N}\} \\
&= \sup\{\|\langle\langle(\pi_\rho)_n(v), x\rangle\rangle\| : x \in \mathcal{B}(\bar{V_\rho}, M_n),\ \|x\| \leq 1\} \\
&= \|(\pi_\rho)_n(v)\|_{\max}
\end{aligned}$$

as in Example 2.2.2. □

Example 3.2.1
Consider Example 3.1.3 of a Lie group G acting on an operator space V via σ. We observed that the matrix seminorms $\boldsymbol{\gamma}^{\boldsymbol{k}}$ are an ordered family which generate the matrix topology on V_σ^∞. So we have that

$$V_\sigma^\infty = \varprojlim (V_\sigma^\infty)^-_{\gamma^k} = \varprojlim V^k$$

where V^k is the operator space of elements of V which are k-times differentiable (ie. D^μ exists for all $|\mu| \leq k$), with the matrix norm $\boldsymbol{\gamma}^{\boldsymbol{k}}|_{V^k}$—that this is a matrix norm follows since $\boldsymbol{\gamma}^{\boldsymbol{k}} \geq (\|\cdot\|_n)$.

If V is instead a local operator space $V = \varprojlim V_\rho$, where the $\rho \in \mathcal{R}$ are ordered and generate the matrix topology, then

$$\boldsymbol{\gamma}^{\boldsymbol{\rho},\boldsymbol{k}} = \sup_{\substack{\boldsymbol{\rho}' \in \mathfrak{R} \\ \boldsymbol{\rho}' \leq \boldsymbol{\rho}}} \sum_{\|\mu\| \leq k} \frac{\rho'_n(D_n^\mu(v))}{\mu!}$$

for $\boldsymbol{\rho} \in \mathfrak{R}$, and so

$$V_\sigma^\infty = \varprojlim_{\boldsymbol{\rho} \in \mathfrak{R}, k \in \mathbb{N}} V_{\gamma^{\rho,k}} = \varprojlim_{\boldsymbol{\rho} \in \mathfrak{R}, k \in \mathbb{N}} V_\rho^k$$

where we consider G acting on V_ρ via the natural action induced by π_ρ.

Example 3.2.2
N.C. Phillips introduced *pro-C*-algebras* in [29]. They are defined to be projective limits of C*-algebras A_ρ where the connecting maps $\pi_{\rho,\gamma}$ are *-homomorphisms. Since *-homomorphisms of C*-algebras are automatically completely bounded, we get immediately that a pro-C*-algebra A is a local operator space, and that it can be written as

$$A = \varprojlim A_\rho.$$

Example 3.2.3
Looking more closely at Example 3.1.4, we see that what we have done is to express $\mathcal{S}(\mathbb{N}^d)$ as a projective limit in two ways:

$$\mathcal{S}(\mathbb{N}^d) = \varprojlim \ell^\infty$$

where the projection maps act via

$$\pi_k : (v_n) \mapsto ((d + |n|)^k v_n),$$

and

$$\mathcal{S}(\mathbb{N}^d) = \varprojlim \ell^2$$

where the projection maps act via

$$\theta_k : (v_n) \mapsto ((d + |n|)^{k/2} v_n).$$

Then the quantizations discussed in the example are just $\mathcal{S}(\mathbb{N}^d) = \varprojlim \ell^\infty$ with the same connecting maps, regarding ℓ^∞ as a C*-algebra, and hence an operator space; and $\mathcal{S}(\mathbb{N}^d) = \varprojlim \ell^2_r$ with the same connecting maps. Clearly this second quantization opens other possibilities, for example $\mathcal{S}(\mathbb{N}^d) = \varprojlim \ell^2_c$ or $\mathcal{S}(\mathbb{N}^d) = \varprojlim O(\ell^2)$ where $O(H)$ for H a Hilbert space is Pisier's operator Hilbert space.

Example 3.2.4
Given an O-family A acting on a Hilbert space H with domain $\mathcal{D}$, and assume that A is partially ordered by the ordering of the matrix norms $\|\cdot\|_a$. Then we let

$$A(I) = \{a \in A : \|\cdot\| \leq \|\cdot\|_a\}.$$

and observe that $A(I)$ is ordered by the ordering of the matrix norms. We note that for $a \in A(I)$, $\mathcal{D}(\bar{a})$ is a Hilbert space with inner product

$$\langle\cdot|\cdot\rangle_{\bar{a}} = \langle \bar{a}\cdot|\bar{a}\cdot\rangle.$$

We will denote this Hilbert space as H_a. Since if $a \leq b$, $H_b \subseteq H_a$, we get a projective system and can consider

$$\overline{\mathcal{D}_A} = \varprojlim{}_{A(I)} H_a$$

where the connecting maps are the embeddings. If $\mathcal{D}$ was maximal in the sense that

$$\mathcal{D} = \bigcap_{a \in A} \mathcal{D}(\bar{a})$$

then $\mathcal{D}_A = \overline{\mathcal{D}_A}$ and the topologies agree ([36], Chapter 2).

So assume that $\mathcal{D}$ is maximal in this sense, then we can extend this result to say that the $\mathcal{D}_{A,c}$ of Example 3.1.5 is a projective limit of Hilbert column spaces

$$\mathcal{D}_{A,c} = \varprojlim (H_a)_c$$

since the inclusions are bounded maps $H_b \to H_a$, and so are in $\mathcal{CB}((H_b)_c, (H_a)_c)$.

3.3 Tensor Products

The two tensor products of primary interest in the theory of locally convex spaces are the projective and injective tensor products, which we will denote $V \widehat{\otimes} W$ and $V \check{\otimes} W$ respectively for V, W locally convex spaces. We have that if $V = \varprojlim V_\rho$, $W = \varprojlim W_\sigma$ (assuming the V_ρ and W_σ are Banach spaces), then we get the incomplete projective and injective tensor products

$$V \underset{\wedge}{\otimes} W = \varprojlim V_\rho \widehat{\otimes} W_\sigma$$
$$V \underset{\vee}{\otimes} W = \varprojlim V_\rho \check{\otimes} W_\sigma .$$

These products are usually denoted $V \otimes_\pi W$ and $V \otimes_\varepsilon W$ respectively in the literature on locally convex spaces. We will use this non-standard notation to highlight the similarities with the Banach space and operator space tensor products. The complete projective and injective products are simply the completions of $V \underset{\wedge}{\otimes} W$ and $V \underset{\vee}{\otimes} W$ respectively.

It is well-known that there are analogues of the projective and injective tensor products in the category of operator spaces. Recall that if V and W are operator spaces, then we define the projective or maximal tensor product of V and W to be the vector space $V \otimes W$ with the matrix norm given by

$$\|u\|_\wedge = \inf\{\|\sigma\|_{cb}\|v\|_p\|w\|_q : u = \sigma \cdot_{p\times q} (v \otimes w),$$
$$v \in M_p(V),\ w \in M_q(W),\ \sigma \in \mathcal{CB}(M_{p\times q}, M_n)\}$$

for any $u \in M_n(V \otimes W)$; and the injective of minimal tensor product to be $V \otimes W$ with the matrix norm

$$\|u\|_\vee = \sup\{\|(\varphi \otimes \psi)_n(u)\| : \varphi \in M_p(V^*), \psi \in M_q(W^*), \|\varphi\|, \|\psi\| \leq 1\}$$

for any $u \in M_n(V \otimes W)$. We denote these spaces and their completions respectively as $V \underset{\wedge}{\otimes}_{\mathrm{op}} W$, $V \underset{\vee}{\otimes}_{\mathrm{op}} W$, $V \widehat{\otimes}_{\mathrm{op}} W$ and $V \check{\otimes}_{\mathrm{op}} W$.

This tells us immediately by analogy what the "correct" definition of the tensor products must be for local operator spaces.

Definition 3.3.1
Let $V = \varprojlim V_\rho$ and $W = \varprojlim W_\sigma$ be local operator spaces (where the V_ρ and W_σ are operator spaces). Then we define the incomplete projective (or maximal) and injective (or minimal) tensor products of V and W to be the matrix topologies on $V \otimes W$ given by

$$V \underset{\wedge}{\otimes}_{op} W = \varprojlim V_\rho \widehat{\otimes}_{op} W_\sigma$$
$$V \underset{\vee}{\otimes}_{op} W = \varprojlim V_\rho \check{\otimes}_{op} W_\sigma .$$

We define the complete projective *(or* maximal*) and* injective *(or* minimal*)* tensor products *of V and W to be the completions of $V \underset{\frown}{\otimes}_{op} W$ and $V \underset{\smile}{\otimes}_{op} W$ respectively and denote them $V \widehat{\otimes}_{op} W$ $V \breve{\otimes}_{op} W$.*

Alternatively, we can construct these spaces by giving $V \otimes W$ the matrix topology coming from the matrix seminorms

$$\boldsymbol{\rho} \widehat{\otimes}_{\mathrm{op}} \boldsymbol{\sigma}(u) = \inf\{\|\sigma\|_{cb}\rho_p(v)\sigma_q(w) : \\ u = \sigma \cdot_{p\times q} (v \otimes w),\ v \in M_p(V),\ w \in M_q(W),\ \sigma \in \mathcal{CB}(M_{p\times q}, M_n)\}$$

and

$$\boldsymbol{\rho} \breve{\otimes}_{\mathrm{op}} \boldsymbol{\sigma}(u) = \sup\{\|(\varphi \circ \pi_\rho \otimes \psi \circ \pi_\sigma)_n(u)\| : \\ \varphi \in M_p(V_\rho^*), \psi \in M_q(W_\sigma^*), \|\varphi\|, \|\psi\| \leq 1\}$$

respectively, for $\boldsymbol{\rho} \in \mathfrak{N}(V)$ and $\boldsymbol{\sigma} \in \mathfrak{N}(W)$.

Furthermore, these tensor products have the properties that you would expect from consideration of the classical and operator space tensor products: they are symmetric, associative and functorial.

Lemma 3.3.1
Let V and W be local operator spaces. Then

$$V \underset{\frown}{\otimes}_{op} W \cong W \underset{\frown}{\otimes}_{op} V \tag{3.3}$$

$$V \underset{\smile}{\otimes}_{op} W \cong W \underset{\smile}{\otimes}_{op} V \tag{3.4}$$

and if X is also a local operator space, then

$$(V \underset{\frown}{\otimes}_{op} W) \underset{\frown}{\otimes}_{op} X \cong V \underset{\frown}{\otimes}_{op} (W \underset{\frown}{\otimes}_{op} X) \tag{3.5}$$

$$(V \underset{\smile}{\otimes}_{op} W) \underset{\smile}{\otimes}_{op} X \cong V \underset{\smile}{\otimes}_{op} (W \underset{\smile}{\otimes}_{op} X). \tag{3.6}$$

If $\varphi \in \mathcal{CC}(X, V)$ and $\psi \in \mathcal{CC}(Y, W)$, then

$$\varphi \otimes \psi \in \mathcal{CC}(X \underset{\frown}{\otimes}_{op} Y, V \underset{\frown}{\otimes}_{op} W)$$
$$\varphi \otimes \psi \in \mathcal{CC}(X \underset{\smile}{\otimes}_{op} Y, V \underset{\smile}{\otimes}_{op} W).$$

Proof :

These all follow immediately from the corresponding statements for operator spaces. If we have $V = \varprojlim V_\rho$, $W = \varprojlim W_\sigma$, $X = \varprojlim X_\gamma$ and $Y = \varprojlim Y_\delta$, then since

$$\begin{aligned} V_\rho \underset{\frown}{\otimes}_{\mathrm{op}} W_\sigma &\cong W_\sigma \underset{\frown}{\otimes}_{\mathrm{op}} V_\rho \\ V_\rho \underset{\smile}{\otimes}_{\mathrm{op}} W_\sigma &\cong W_\sigma \underset{\smile}{\otimes}_{\mathrm{op}} V_\rho \end{aligned}$$

we immediately get (3.3) and (3.4); since

$$(V_\rho \underset{\frown}{\otimes}_{\mathrm{op}} W_\sigma) \underset{\frown}{\otimes}_{\mathrm{op}} X_\gamma \cong V_\rho \underset{\frown}{\otimes}_{\mathrm{op}} (W_\sigma \underset{\frown}{\otimes}_{\mathrm{op}} X_\gamma)$$
$$(V_\rho \underset{\smile}{\otimes}_{\mathrm{op}} W_\sigma) \underset{\smile}{\otimes}_{\mathrm{op}} X_\gamma \cong V_\rho \underset{\smile}{\otimes}_{\mathrm{op}} (W_\sigma \underset{\smile}{\otimes}_{\mathrm{op}} X_\gamma)$$

we immediately get (3.5) and (3.6). Finally, given any $\boldsymbol{\rho}$ and $\boldsymbol{\sigma}$, we can find $\boldsymbol{\gamma}$, $\boldsymbol{\delta}$ and $k > 0$ so that

$$\boldsymbol{\gamma}(\boldsymbol{\varphi}_n(v)) \leq k\boldsymbol{\rho}(v)$$

and

$$\boldsymbol{\delta}(\boldsymbol{\psi}_n(w)) \leq k\boldsymbol{\sigma}(w)$$

and then we have that

$$(\boldsymbol{\gamma} \widehat{\otimes}_{\mathrm{op}} \boldsymbol{\delta})((\boldsymbol{\varphi} \otimes \boldsymbol{\psi})(z)) \leq k(\boldsymbol{\rho} \widehat{\otimes}_{\mathrm{op}} \boldsymbol{\sigma})(z)$$

and

$$(\boldsymbol{\gamma} \check{\otimes}_{\mathrm{op}} \boldsymbol{\delta})((\boldsymbol{\varphi} \otimes \boldsymbol{\psi})(z)) \leq k(\boldsymbol{\rho} \check{\otimes}_{\mathrm{op}} \boldsymbol{\sigma})(z)$$

from the operator space results and so we get matrix continuity for both projective and injective tensor products. □

The last part of this lemma immediately implies that the matrix topologies given by the constructions do not depend upon the choice of generating family used in the projective limits.

We also expect the projective and injective tensor products to be injective and projective.

Lemma 3.3.2
Let V, W and X and Y be local operator spaces. If $\varphi : V \to X$ and $\psi : W \to Y$ are matrix quotient mappings, then

$$\varphi \otimes \psi : V \underset{\smile}{\otimes}_{op} W \to X \underset{\smile}{\otimes}_{op} Y$$

is a matrix quotient mapping. If, on the other hand, φ and ψ are inclusions of local operator spaces, then

$$\varphi \otimes \psi : V \underset{\frown}{\otimes}_{op} W \to X \underset{\frown}{\otimes}_{op} Y$$

is an inclusion of local operator spaces.

The proof of this lemma is the same as the proof of Lemma 3.3.1: we simply lift the result from the corresponding result for operator spaces.

We can identify the operator space tensor products with certain mapping spaces. At present we do not have any natural topologies to apply to mapping spaces of local operator spaces, never the less we can make the following identification of vector spaces.

Lemma 3.3.3
Let V, W and X be local operator spaces. We say that a bilinear map $\varphi : V \times W \to X$ is matrix continuous *if for every matrix seminorm $\boldsymbol{\gamma} \in \mathfrak{N}(X)$ we have matrix seminorms $\boldsymbol{\rho} \in \mathfrak{N}(V)$ and $\boldsymbol{\sigma} \in \mathfrak{N}(W)$ such that*

$$\gamma_{n\times m}(\varphi_{n;m}(v, w)) \leq \rho_n(v)\sigma_m(w)$$

for any n, $m \in \mathbb{N}$, $v \in M_n(V)$, $w \in M_n(W)$. We denote the space of all such maps by $\mathcal{CC}(V \times W, X)$.

Then $\mathcal{CC}(V \underset{\smile}{\otimes}_{op} W, X) \cong \mathcal{CC}(V \times W, X)$ as vector spaces, via

$$\psi_{n\times m}(v \otimes w) \leftrightarrow \psi_{n;m}(v, w).$$

Proof :
If $\psi \in \mathcal{CC}(V \times W, X)$, then given $\boldsymbol{\gamma} \in \mathfrak{N}(X)$, ψ induces a contractive map

$$\bar{\psi} : \bar{V}_{\boldsymbol{\rho}} \times \bar{W}_{\boldsymbol{\sigma}} \to \bar{X}_{\boldsymbol{\gamma}}$$

and so $\bar{\psi} \in \mathcal{CB}(\bar{V}_{\boldsymbol{\rho}} \widehat{\otimes}_{\mathrm{op}} \bar{W}_{\boldsymbol{\sigma}}, \bar{X}_\gamma)$ and so $\psi \in \mathcal{CC}(V \underset{\smile}{\otimes}_{\mathrm{op}} W, X)$.

Conversely, if $\psi \in \mathcal{CC}(V \widehat{\otimes}_{\mathrm{op}} W, X)$, then for every $\boldsymbol{\gamma} \in \mathfrak{N}(X)$, ψ induces a contraction

$$\bar{\psi} : \bar{V}_{\boldsymbol{\rho}} \widehat{\otimes}_{\mathrm{op}} \bar{W}_{\boldsymbol{\sigma}} \to \bar{X}_{\boldsymbol{\gamma}}$$

for some $\boldsymbol{\rho} \in \mathfrak{N}(V)$ and $\boldsymbol{\sigma} \in \mathfrak{N}(W)$ and so $\bar{\psi} \in \mathcal{CB}(\bar{V}_{\boldsymbol{\rho}} \times \bar{W}_{\boldsymbol{\sigma}}, \bar{X}_\gamma)$. But this implies that

$$\gamma_{n\times m}(\varphi_{n;m}(v, w)) \leq \rho_n(v)\sigma_m(w)$$

for any n, $m \in \mathbb{N}$, $v \in M_n(V)$, $w \in M_n(W)$. □

Finally, we have that for V and W operator spaces, the identity map extends to a complete contraction

$$V \widehat{\otimes}_{\mathrm{op}} W \to V \check{\otimes}_{\mathrm{op}} W.$$

Lemma 3.3.4
Let V and W be local operator spaces. Then the identity map

$$V \underset{\smile}{\otimes}_{op} W \to V \otimes_{op} W$$

is matrix continuous.

Proof :
We simply note that given $\boldsymbol{\rho} \in \mathfrak{N}(V)$ and $\sigma \in \mathfrak{N}(W)$, we have that for any $x \in M_n(V \otimes W)$

$$\boldsymbol{\rho} \check{\otimes}_{\text{op}} \boldsymbol{\sigma}(x) \leq \boldsymbol{\rho} \hat{\otimes}_{\text{op}} \boldsymbol{\sigma}(x).$$

□

The theory of operator spaces has an additional, novel and natural tensor product, the Haagerup tensor product. To define the Haagerup tensor product we introduce a new operation $\odot$ where

$$v \odot w = [\sum_{k=1}^{q} v_{j,k} \otimes w_{k,l}]$$

where $v \in M_{p,q}(V)$, $w \in M_{q,r}(V)$ and $v \odot w \in M_{p,r}(V \otimes W)$ (V and W are arbitrary vector spaces at this point). In other words the operator $\odot$ is a combination of matrix multiplication and tensor product. We define the (incomplete) Haagerup tensor product of two operator spaces V and W to be the $V \otimes W$ with the matrix norm

$$||u||_h = \inf\{||v||||w|| : u = v \odot w, v \in M_{n,r}(V), w \in M_{r,n}(W)\}$$

for any $u \in M_n(V \otimes W)$. We denote this space to be $V \otimes_h W$, and the completion to be $V \otimes^h W$. The Haagerup tensor product is associative and functorial, but is not symmetric. It lies between the injective and projective tensor products in the sense that we have complete contractions

$$V \hat{\otimes}_{\text{op}} W \to V \otimes^h W \to V \check{\otimes}_{\text{op}} W$$

induced by the identity map on $V \otimes W$. It also has the property that it linearizes certain multilinear mappings: if

$$\varphi : V_1 \times V_2 \times \ldots \times V_n \to W$$

is a multiplicatively bounded multilinear map, that is to say

$$||\boldsymbol{\varphi}||_{mb} = \sup\{||\boldsymbol{\varphi}(v_1, \ldots, v_n)|| : ||v_i|| \leq 1\}$$

where $v_i \in M_{p_{i-1},p_i}(V_i)$, $p_i \in \mathbb{N}$, then φ extends to a completely bounded linear map (also called φ by abusing notation slightly)

$$\varphi : V_1 \otimes_h V_2 \otimes_h \ldots \otimes_h V_n \to W.$$

The converse statement is also the case. These fact will be key later on, since it in particular implies that if A is an operator algebra, then the multiplication map is a completely bounded map from $A \otimes_h A \to A$.

The Haagerup tensor product naturally extends to the category of local operator spaces via projective limits.

Definition 3.3.2
Let $V = \varprojlim V_\rho$ and $W = \varprojlim W_\sigma$ be local operator spaces (where the V_ρ and W_σ are operator spaces). Then we define the incomplete Haagerup tensor product *of V and W to be $V \otimes W$ with the matrix topology given by*

$$V \otimes_h W = \varprojlim V_\rho \otimes_h W_\sigma$$

We define the complete Haagerup tensor product *of V and W to be the completion $V \otimes^h W$ of $V \otimes_h W$.*

The Haagerup tensor product of local operator spaces is associative and functorial.

Lemma 3.3.5
If V, W, X and Y are local operator spaces, then

$$(V \otimes_h W) \otimes_h X \cong V \otimes_h (W \otimes_h X)$$

and if $\varphi \in \mathcal{CB}(X, V)$, $\psi \in \mathcal{CB}(Y, W)$, then $\varphi \otimes \psi \in \mathcal{CB}(X \otimes_h Y, V \otimes_h W)$.

It also lies between the injective and projective tensor products.

Lemma 3.3.6
If V and W are local operator spaces, then we have that the identity is a completely continuous map

$$V \underset{\smile}{\otimes}_{op} W \to V \otimes_h W$$

and

$$V \otimes_h W \to V \underset{\frown}{\otimes}_{op} W$$

The proof of these lemmas is identical to the appropriate parts of Lemma 3.3.1 and Lemma 3.3.4, and so are omitted. Once again we get as an immediate corollary that the Haagerup tensor product does not depend upon the way that we chose to write V and W as projective limits of operator spaces. The Haagerup tensor product is not symmetric.

The Haagerup tensor product also inherits the property of linearizing multiplicatively multilinear maps. If $V_1, \ldots, V_n$ and W are local operator spaces, then a multilinear map

$$\varphi : V_1 \times \ldots \times V_n \to W$$

is multiplicatively continuous if for every $\boldsymbol{\sigma} \in \mathfrak{N}(W)$, there exist $\boldsymbol{\rho_i} \in \mathfrak{N}(V_i)$ such that

$$\boldsymbol{\sigma}(\boldsymbol{\varphi}(v_1, \ldots, v_n)) \leq \boldsymbol{\rho_i}(v_1) \ldots \boldsymbol{\rho_n}(v_n)$$

for $v_i \in M_{p_{i-1}, p_i}(V_i)$.

Lemma 3.3.7
If $\varphi : V_1 \times \ldots \times V_n \to W$ is multiplicatively continuous, then it extends to a matrix continuous map (which we will also call φ) from $V_1 \otimes_h \ldots \otimes_h V_n$ to W.

Again, this result follows by lifting the corresponding result up from the operator space case.

Example 3.3.1
Let A be a C*-algebra with a strongly continuous action σ of a Lie group G. Then we have a local operator space structure A_σ^∞ on the $\mathcal{C}^\infty$ vectors in A as in Example 3.1.3. Consider seminorms of the form $\boldsymbol{\gamma^k}$ given by

$$\gamma_n^k(a) = \sum_{|\mu| \leq k} \frac{\|D_n^\mu(a)\|}{\mu!}$$

where $\mu! = \mu_1!\mu_2!\ldots\mu_q!$. For any $a \in M_n(A_\sigma^\infty \otimes_h A_\sigma^\infty)$, and any expression of $a = u \odot v$, we have

$$\begin{aligned}
\gamma_n^k(uv) &= \sum_{|\mu|\leq k} \frac{\|D_n^\mu([\sum_l u_{i,l}v_{l,j}])\|}{\mu!} \\
&= \sum_{|\mu|\leq k} \frac{\|\sum_l D^\mu(u_{i,l}v_{l,j})\|}{\mu!} \\
&= \sum_{l,|\mu|\leq k} \frac{\|[\sum_{l,\nu\leq\mu} C(\mu,\nu)D^\nu(u_{i,l})D^{\mu-\nu}(v_{l,j})]\|}{\mu!} \\
&\leq \sum_l \left(\sum_{|\nu|\leq k} \frac{\|[D^\nu(u_{i,j})]\|}{\nu!}\right)\left(\sum_{|\eta|\leq k} \frac{\|[D^\eta(v_{i,j})]\|}{\eta!}\right) \\
&\leq \gamma_{n,r}^k(u)\gamma_{r,n}^k(v)
\end{aligned}$$

since multiplication is completely contractive from $A \otimes^h A \to A$, and where

$$C(\mu,\nu) = C(\mu_1,\nu_1)C(\mu_2,\nu_2)\ldots C(\mu_n,\nu_n) = \frac{\mu!}{\nu!(\mu-\nu)!}$$

and so taking infima of both sides we get

$$\boldsymbol{\gamma}^{\boldsymbol{k}}(m(a)) \leq \boldsymbol{\gamma}^{\boldsymbol{k}} \otimes^h \boldsymbol{\gamma}^{\boldsymbol{k}}(a)$$

and so the multiplication map is matrix continuous as a map

$$m : A_\sigma^\infty \otimes_h A_\sigma^\infty \to A_\sigma^\infty.$$

This result still holds if we replace A by an abstract operator algebra.

Example 3.3.2
Continuing Example 3.1.6, we have that the space of physical states is given by

$$T = W \widehat{\otimes} W$$

and is the dual of the algebra of observables [6]. The action of T on $\mathcal{A}$ is implemented by

$$\langle v \otimes w, a\rangle = \langle Jv, aw\rangle$$

where we are using the physicist's inner product (ie. antilinear in the first position) and $J : H \to H$ is the standard antilinear involution induced by the isomorphism of H with its conjugate Hilbert space $\bar{H}$.

Given that W_c in fact fits Example 3.2.4, we have that

$$W_c = \varprojlim(H_a)_c$$

and that what we are in fact doing is considering the first W in the tensor product as row vectors, it makes sense to consider

$$W_r = \varprojlim(H_a)_r.$$

We also observe that what we are doing is analogous to the operator space fact that $\mathcal{T}(K) = K_r \otimes^h K_c$ is the dual of $\mathcal{K}(K) = K_c \otimes^h K_r$. So a natural local operator space version of T, the *physical matrix states*, is

$$W_r \otimes^h W_c.$$

3.4 Complete Boundedness and Mapping Spaces

If V, W are locally convex spaces, then the space of continuous linear maps $\mathcal{C}(V,W)$ is clearly a vector space, but it is not immediately clear what topology we should put on it, or even if there is one. We say that a set $X \in V$ is *bounded* if for every seminorm $\rho \in \mathfrak{N}(V)$, there is a $\lambda > 0$ such that $\lambda X \subseteq B^\rho$. If $\varphi : V \to W$ is a linear map, we say that φ is *bounded* and write $\varphi \in \mathcal{B}(V,W)$ if φ maps bounded sets in V to bounded sets in W. If X is bounded, closed and convex we have an associated gauge $p = \gamma^X$ which we call a bounded gauge. We observe that p is faithful, and so ${}_pV$ is a normed vector space.

Letting $\mathfrak{S}$ be a family of bounded gauges such that there is no $v \in V$ with $\rho(v) = 0$ for all $\rho \in \mathfrak{S}$, we define a topology on $\mathcal{C}(V,W)$,

$$\mathcal{C}(V,W)_{\mathfrak{S}} = \varprojlim\{(\mathcal{B}({}_pV, W_\rho), \pi_{p,\rho}) : p \in \mathfrak{S}, \rho \in \mathfrak{N}(W)\}$$

where $\pi_{p,\rho}$ is the map which takes $\varphi \in \mathcal{C}(V,W)$ to the map obtained by the composition

$${}_pV \longrightarrow V \xrightarrow{\varphi} W \longrightarrow W_\rho.$$

Classically there are a number of standard choices for $\mathfrak{S}$.

- $\mathfrak{S}$ contains only the Minkowski seminorms of closed convex hulls of finite sets. Then we get the topology of pointwise convergence.
- $\mathfrak{S}$ contains only the Minkowski seminorms of convex, compact sets.
- $\mathfrak{S}$ contains all bounded seminorms.

As the reader would expect by this stage, the above discussion generalizes to the case of local operator spaces.

Definition 3.4.1
Let $\boldsymbol{X} = (X_n)$ be a matrix subset of a local operator space V. We say that $\boldsymbol{X}$ is completely bounded *if for every matrix seminorm $\boldsymbol{\rho} \in \boldsymbol{\mathfrak{N}}(V)$ there is a $\lambda > 0$ such that $\lambda\boldsymbol{X} \subseteq \boldsymbol{B^\rho}$.*

Given a linear map $\varphi : V \to W$, we say that it is *completely bounded* and write $\varphi \in \mathcal{CB}(V,W)$ if φ maps completely bounded matrix sets in V to completely bounded matrix sets in W. If $\boldsymbol{X}$ is matrix convex and closed, it has an associated Minkowski matrix gauge $\boldsymbol{\rho^X}$, which is faithful, so ${}_{\boldsymbol{\rho^X}}V$ is an operator space. We call such matrix gauges whose matrix unit sets are completely bounded, completely bounded.

Definition 3.4.2
Let V, W be local operator spaces. Given a family $\boldsymbol{\mathfrak{S}}$ of completely bounded matrix gauges on V such that there is no $v \in V$ with $\rho_1(v) = 0$ for all $\boldsymbol{\rho} \in \boldsymbol{\mathfrak{S}}$, we define the $\boldsymbol{\mathfrak{S}}$ matrix topology on $\mathcal{CC}(V,W)$ to be given by

$$\mathcal{CC}(V,W)_{\mathfrak{S}} = \boldsymbol{\varprojlim}\{(\mathcal{CB}({}_{\boldsymbol{p}}V, W_{\boldsymbol{\rho}}), \pi_{\boldsymbol{p},\boldsymbol{\rho}}) : \boldsymbol{p} \in \boldsymbol{\mathfrak{S}}, \boldsymbol{\rho} \in \boldsymbol{\mathfrak{N}}(W)\}$$

where the maps $\pi_{\boldsymbol{p},\boldsymbol{\rho}}$ are the maps which take $\varphi \in \mathcal{C}(V,W)$ to the map obtained by the composition

$${}_pV \longrightarrow V \xrightarrow{\varphi} W \longrightarrow W_\rho.$$

Example 3.4.1
If $\boldsymbol{\mathfrak{S}}$ is the collection of all the completely bounded matrix gauges $\boldsymbol{\mathfrak{B}}(V)$ on V, then we call this matrix topology the *completely bounded matrix topology.* In the special case where $W = \mathbb{C}$, this is the *strong matrix topology.*

Example 3.4.2
If we look at matrix sets which are the closed (absolutely) matrix convex hulls of points $\boldsymbol{X_v}$ where $(X_v)_m = \varnothing$ if $n \neq m$ and $(X_v)_n = \{v\}$, and consider their Minkowski matrix gauges $\boldsymbol{\rho^v}$, then this family of bounded matrix gauges $\mathfrak{W}(V)$ gives what we will call the *matrix pointwise topology*. In the special case where $W = \mathbb{C}$, this is the *weak matrix topology*.

Clearly the matrix pointwise topology is weaker than the completely bounded matrix topology, since the identity map

$$\mathrm{id} : \mathcal{CC}(V,W)_{\mathfrak{B}(V)} \to \mathcal{CC}(V,W)_{\mathfrak{W}(V)}$$

is clearly matrix continuous. Indeed if $\mathfrak{S}$ determines a matrix topology on $\mathcal{CC}(V,W)$, then it is always weaker than the completely bounded matrix topology, since $\mathfrak{S} \subseteq \mathfrak{B}(V)$. It is also always stronger than the matrix pointwise topology, since if $v \in M_n(V)$, then there is a $\boldsymbol{\rho} \in \mathfrak{S}$ with $k = \rho_n(v) > 0$ and moreover the matrix convex hull of v must be contained in $k\boldsymbol{B^\rho}$, and so for any $w \in V$,

$$\boldsymbol{\rho}(w) \leq k^{-1}\boldsymbol{\rho^v}(w).$$

We would like to construct a matrix version of the second of the standard topologies. The difficulty is that exactly we mean by a “compact” matrix set is problematical, as we will see in the next chapter. However in this context we will say that a matrix convex set $\boldsymbol{X} \subseteq \boldsymbol{V}$ is *matrix compact* if each of the X_n is compact in $M_n(V)$.

Example 3.4.3
Let W be the space of wave functions. Dubin and Hennings [6] identify the algebra of observables $\mathcal{A}$ with $\mathcal{L}^+(W,W)$, the space of unbounded closable operators on H, with domain W and range contained in W, and with the property that the domains of the adjoints a^+ contain W. Hence the restrictions of the adjoints $a^* = a^+|_W$ are in $\mathcal{L}^+(W,W)$. Also $\mathcal{L}^+(W,W)$ is clearly a *-algebra under composition, indeed it is a prototypical *O*-algebra*. They topologize $\mathcal{A}$ by identifying it with a subspace of $\mathcal{C}_{\mathfrak{B}}(W, W'_{\mathfrak{B}})$ in the obvious way.

Considering W_c, we can instead regard the algebra of observables as $\mathcal{A} = \mathcal{CC}(W_c, W_c)$ sitting inside $\mathcal{CC}_{\mathfrak{B}}(W_c, (W'_c)_{\mathfrak{B}})$. Unwinding all this notation, the matrix topology we get on $\mathcal{A}$ is that generated by the matrix seminorms

$$\boldsymbol{p_{P,Q}}([a_{i,j}]) = \sup_{\xi \in P_n, \eta \in Q_n} \|[\langle \sum_k a_{i,k}\xi_{k,j}, \eta_{i,j}\rangle]\|.$$

where $\boldsymbol{P}$ and $\boldsymbol{Q}$ are completely bounded subsets of W_c.

3.5 The Uniform Complete Boundedness Principle

We now move in the direction of proving a uniform boundedness principle for local operator spaces. Classically, the uniform boundedness principle holds if and only if the locally convex space is barrelled; a condition which says that every closed, (absolutely) convex, absorbent set is a neighborhood.

Definition 3.5.1
A matrix subset $\boldsymbol{X}$ *of a vector space* V *is* completely absorbent *if for every* $v \in M_n(V)$*, there is a* $\lambda > 0$ *such that* $v \in \alpha\boldsymbol{X}\beta$ *for all* $\alpha \in M_{n,m}$*,* $\beta \in M_{m,n}$ *with* $\|\alpha\|\|\beta\| > \lambda$*.*

A matrix subset $\boldsymbol{X}$ *of a local operator space* V *is called* a matrix barrel *if it is absolutely matrix convex, completely absorbent, and closed.*

V *is a* matrix barrelled *local operator space if every barrel is a matrix neighborhood of 0, that is it is a matrix set containing the matrix unit set of a continuous matrix seminorm.*

Notice that if $\boldsymbol{X}$ is matrix convex then $\boldsymbol{X}$ is completely absorbent if and only if for every $v \in M_n(V)$ there is a $\lambda > 0$ such that $v \in \mu\boldsymbol{X}$ for all $\mu > \lambda$, that is it is absorbent at each level. To see this we simply observe that if $v = \alpha x \beta$ for some $x \in \boldsymbol{X}$ and $\mu = \|\alpha\|\|\beta\| > \lambda$, then

$$x' = \alpha x \beta / \mu \in \boldsymbol{X}$$

by matrix convexity and $v = \mu x'$. The converse is trivial: we take $\alpha = \mu I$, $\beta = I$ to find the x we need.

Given this fact the fact that matrix barrelled spaces are barrelled locally convex spaces is trivial. In the other direction we observe that if V is barrelled and we have a matrix barrel in $\max V$, then it contains some unit set of a continuous seminorm at the first level, and so it contains the minimal quantization of that set and so contains a matrix unit set of $\max V$.

However if V is any distinct quantization other than $\max V$, we have that there must be some continuous matrix seminorm $\hat{\boldsymbol{\rho}}$ on $\max V$ such that for all $\boldsymbol{\sigma} \in \mathfrak{N}(V)$ we have at some point $v_{\boldsymbol{\sigma}} \in M_n(V)$

$$\boldsymbol{\rho}(v_{\boldsymbol{\sigma}}) > \boldsymbol{\sigma}(v_{\boldsymbol{\sigma}}).$$

Hence the unit set of $\hat{\boldsymbol{\rho}}$ is a matrix barrel, but does not contain any matrix unit set of V, and so is not a neighborhood.

Due to the highly restricted nature of the spaces which are matrix barrelled, it is not clear if this is the "correct" definition. Nevertheless, in the case of operator nuclear spaces, where all quantizations are the same, we will see that this definition of a barrelled space is what we need. The other cue that this definition is good, is that it does allow us to prove an analogue of the uniform boundedness principle.

Recall that the classical principle of uniform boundedness says that if we have a barrelled space V and a locally convex space W, then a set of continuous linear mappings from V to W which is pointwise bounded is equicontinuous.

Let $\boldsymbol{X}$ be a matrix subset of $\mathcal{CC}(V,W)$. We will say that $\boldsymbol{X}$ is *matrix pointwise bounded* if the matrix set $\boldsymbol{X}(v)$ where

$$X(v)_n = \{\varphi_k(v) : \varphi \in X_n\})$$

is a completely bounded subset of $M_n(W)$ for every $v \in M_n(V)$. We will say $\boldsymbol{X}$ is *matrix equicontinuous* if given any continuous matrix seminorm $\boldsymbol{\sigma}$ on W, there is a continuous matrix seminorm $\boldsymbol{\rho}$ on V such that

$$\sigma_{mn}(\varphi_n(v)) \leq \rho_n(v)$$

for all m, $\varphi \in X_m$ and $v \in M_n(V)$. For our purposes it will help to rephrase this in terms of matrix unit sets: if $\boldsymbol{U}$ is a matrix unit set in W, then we can find a matrix unit set $\boldsymbol{Z}$ of V such that $\boldsymbol{\varphi}(\boldsymbol{Z}) \subseteq \boldsymbol{U}$. As is ideal in the theory of operator spaces, we have set things up so that we can prove the following result in an entirely straightforward manner.

Theorem 3.5.1 (Principle of Uniform Complete Boundedness)
Let V be a matrix barrelled space, W a local operator space. If $\boldsymbol{X}$ is a matrix subset of $\mathcal{CC}(V,W)$ which is matrix pointwise bounded, then is matrix equicontinuous.

Proof :
Choose a closed matrix unit set, $\boldsymbol{U}$, of a continuous matrix seminorm on W. Let $\boldsymbol{B} = (B_n)$ where

$$B_n = \{v \in M_n(V) : \varphi_n(v) \in U_{mn} \text{ for some } m \text{ and } \varphi \in X_m\}.$$

Now we see immediately that $\boldsymbol{B}$ is closed and matrix convex. We would like to show it is a matrix barrel, for if it is then it is a matrix neighborhood since V is matrix barrelled, and so we can find some continuous matrix seminorm whose matrix unit set lies inside $\boldsymbol{B}$, as then

$$\boldsymbol{\varphi}(\boldsymbol{Z}) \subseteq \boldsymbol{U}$$

for every $\varphi \in \boldsymbol{X}$.

Given $v \in M_n(V)$, we have that $\boldsymbol{X}(v)$ is completely bounded, and so there is a $\lambda > 0$ such that $\boldsymbol{X}(v) \subseteq \lambda \boldsymbol{U}$. Moreover, since $\boldsymbol{U}$ is matrix convex, $\boldsymbol{X}(v) \subseteq \mu\boldsymbol{U}$ for any $\mu \geq \lambda$. But what this is saying is that there is a $\varphi \in \boldsymbol{X}$ such that $\varphi_n(\mu^{-1}v) \in \boldsymbol{U}$, so $v \in \mu B_n$, and so $\boldsymbol{B}$ is completely absorbent. □

Example 3.5.1
This gives us a limited analogue of the uniform boundedness principle in the context of operator spaces.

Corollary 3.5.2
Let V be a Banach space, W an operator space and $\boldsymbol{X}$ a matrix subset of $\mathcal{CB}(\max V, W)$ such that

$$\sup_{\varphi \in \boldsymbol{X}} \|\boldsymbol{\varphi}_n(v)\| \leq \infty$$

for all $n \in \mathbb{N}$ and all $v \in M_n(V)$, then

$$\sup_{\varphi \in \boldsymbol{X}} \|\boldsymbol{\varphi}\|_{cb} \leq \infty.$$

3.6 Locally M_∞-Convex Spaces

Recall that we discussed the relationship between matrix convexity and M_∞ convexity in Section 2.4. An obvious question to ask is what notion corresponds to a local operator space in the M_∞ language? And does this viewpoint give us any extra insight?

The correspondence between matrix seminorms and M_∞-module seminorms means that local operator space structures on a vector space V have a one-to-one correspondence with locally M_∞-convex topologies on $M_\infty(V)$ (ie. topological vector space structures on $M_\infty(V)$ which have a neighborhood base at 0 consisting of M_∞-convex sets, or equivalently, the topology is determined by M_∞-module seminorms). We will call such a space a *locally M_∞-convex (topological vector) space.*

In this point of view $\varphi \in \mathcal{CC}(V, W)$ if and only if $\varphi_\infty \in \mathcal{C}(M_\infty(V), M_\infty(W))$. Similarly, if $V = \varprojlim(V_\gamma, \pi_\gamma)$, then $M_\infty(V) = \varprojlim(M_\infty(V_\gamma), (\pi_\gamma)_\infty)$ as a locally convex space. Inclusions and quotients also work just as you would expect: $\varphi : V \hookrightarrow W$ as local operator spaces if and only if $\varphi_\infty : M_\infty(V) \hookrightarrow M_\infty(W)$ as locally convex spaces; $\varphi : V \to W$ is a complete quotient map if and only if $\varphi_\infty : V \to W$ is a quotient map.

One advantage of looking at the theory in this way is that it gives us access to classical results for locally convex spaces in a way which is useful for proving local operator space results. We will use this in Section 3.7 when we discuss bornology. The disadvantage is that we lose access to the large body of results available from operator space theory. For example, the properties of the various tensor products would be tedious to prove using M_∞ methods.

The one other piece of insight that these methods give is that they, somewhat surprisingly, give us information about minimal and maximal quantizations of locally convex spaces. We will use this when we discuss operator nuclear spaces in Section 3.9.

Let V be a Banach space. It is well known (see Effros and Ruan [7], Section 3.4) that

$$M_n(\min V) = M_n \check{\otimes} V.$$

Since M_n is finite dimensional and V is complete, we in fact have that

$$M_n(\min V) = M_n \underset{\smile}{\otimes} V.$$

and so

$$M_\infty(\min V) = M_\infty \underset{\smile}{\otimes} V$$

as normed spaces. On the other hand we have that

$$M_n(\max V) = M_n \otimes V$$

as vector spaces, and the norm on $M_n(V)$ is a cross norm, since if we have $\alpha \in M_n$ and $v \in V$,

$$\begin{aligned}\|\alpha \otimes v\|_{max} &= \sup\{\|\langle\langle \alpha \otimes v, \varphi\rangle\rangle\| : \varphi \in \mathcal{B}(V, M_p), \|\varphi\| \leq 1, p \in \mathbb{N}\}\\ &= \sup\{\|(\mathrm{id} \otimes \varphi)(\alpha \otimes v)\| : \varphi \in \mathcal{B}(V, M_p), \|\varphi\| \leq 1, p \in \mathbb{N}\}\\ &= \sup\{\|\alpha \otimes \varphi(v)\| : f \in \mathcal{B}(V, M_p), \|f\| \leq 1, p \in \mathbb{N}\}\\ &= \sup\{\|\alpha\|\|\varphi(v)\| : f \in \mathcal{B}(V, M_p), \|f\| \leq 1, p \in \mathbb{N}\}\\ &= \|\alpha\|\|v\|.\end{aligned}$$

So there is a contraction

$$M_n \underset{\vee}{\otimes} V \to M_n(\max V)$$

and the standard map $\theta : M_\infty \to M_n : x \mapsto p_n x p_n$ is a contraction, so for any element $x \in M_\infty \underset{\vee}{\otimes} V$ we can factor the map

$$M_\infty \underset{\vee}{\otimes} V \to M_\infty(\max V)$$

through the diagram

$$\begin{array}{ccc} M_n \underset{\vee}{\otimes} V & \longrightarrow & M_n(\max V) \\ \uparrow & & \downarrow \\ M_\infty \underset{\vee}{\otimes} V & \longrightarrow & M_\infty(\max V) \end{array}$$

for n sufficiently large, and so it is a contraction.

If we have that $V = \varprojlim V_\rho$ is now a locally convex space, then

$$M_\infty(\min V) = \varprojlim M_\infty(\min V_\rho) = \varprojlim M_\infty \underset{\vee}{\otimes} V_\rho = M_\infty \underset{\vee}{\otimes} V. \tag{3.7}$$

Also, the map

$$M_\infty \underset{\wedge}{\otimes} V \to M_\infty(\max V) \tag{3.8}$$

must be continuous, since for any ρ, the map

$$M_\infty \underset{\wedge}{\otimes} \bar{V}_\rho \to M_\infty(\max \bar{V}_\rho)$$

is a contraction.

3.7 Injective Limits and Matrix Bornology

Given the importance of projective limits in the material so far, a natural question would be to ask what we can say about injective limits of local operator spaces. Classically, if we are given a vector space V and a family of locally convex spaces V_γ together with mappings $\psi_\gamma : V_\gamma \to V$, the *injective* or *direct limit topology* on V is the finest locally convex topology for which all the ψ_γ are continuous. We write

$$V = \varinjlim V_\gamma = \varinjlim\{(V_\gamma, \psi_\gamma)\}$$

to denote V with this topology. V is a projective limit of such V_γ if and only if the following is true: for any W a locally convex space, $\varphi : V \to W$ is continuous if and only if $\varphi \circ \psi_\gamma$ is continuous for every γ.

If instead we consider local operator spaces V_γ and maps $\psi_\gamma : V_\gamma \to V$, then we say that the *injective* or *direct limit matrix topology* is the finest such that all the ψ_γ are matrix continuous. By the finest matrix topology, we mean the one for which

$$\mathfrak{N}(V) = \{\boldsymbol{\rho} : \forall\ \gamma,\ \boldsymbol{\sigma} \in \mathfrak{N}(V_\gamma)\ \exists\ k_\sigma \text{ such that } \boldsymbol{\rho}(\boldsymbol{\psi_\gamma}(v)) \leq k_\sigma \boldsymbol{\sigma}(v)\}.$$

We denote V with this matrix topology by

$$V = \varinjlim V_\gamma = \varinjlim\{(V_\gamma, \psi_\gamma)\}.$$

This is equivalent to saying that the matrix topology on V corresponds to the injective limit locally convex topology on $M_\infty(V)$ determined by $M_\infty(V_\gamma)$ and $\psi_\infty : M_\infty(V_\gamma) \to M_\infty(V)$.

Again, we can characterize this as saying that V is the matrix injective limit of the V_γ if and only if the following is true: for any W a local operator space, $\varphi : V \to W$ is matrix continuous if and only if $\varphi \circ \psi_\gamma$ is matrix continuous for every γ.

Direct limits are not as useful as projective limits, however they do appear in the classical theory of bornology. We say that a set X in a locally convex space V is *bornivorous* if it absorbs arbitrary bounded sets, that is given any bounded set B, we can find a $\lambda > 0$ such that $\lambda B \subseteq X$. The unit sets of continuous seminorms are bornivorous, by the definition of boundedness. If, conversely, every closed convex bornivorous set is the unit set of continuous seminorm, then we say that the space V is *bornological.*

Proposition 3.7.1
If V is a locally convex space, then the following are equivalent:

i. V is bornological,

ii. for any W locally convex, $\mathcal{B}(V, W) = \mathcal{C}(V, W)$,

iii. $V = \varinjlim\{_pV : p \in \mathfrak{B}(V)\}$.

We say that a matrix set $\boldsymbol{X}$ in a local operator space V is *completely bornivorous* if it absorbs arbitrary completely bounded matrix sets, that is given any completely bounded matrix set $\boldsymbol{B}$ we can find $\lambda > 0$ so that $\lambda\boldsymbol{B} \subseteq \boldsymbol{X}$. Clearly matrix unit sets of continuous matrix seminorms are completely bornivorous. If the converse is true, that is if close matrix convex sets which are completely bornivorous are matrix unit sets of continuous matrix seminorms, then we say that V is *completely bornological.*

We note that we only need to check that $\boldsymbol{X}$ absorbs all matrix convex bounded sets to check if $\boldsymbol{X}$ is bornivorous, since if $\boldsymbol{B}$ is bounded then so is its matrix convex hull. We see that a closed matrix convex set $\boldsymbol{X}$ is completely bornivorous if and only if the corresponding set $X \subseteq M_\infty(V)$ absorbs bounded M_∞-convex sets, since matrix convex bounded sets in V correspond to M_∞-convex bounded sets $M_\infty(V)$. But X will absorb all bounded M_∞-convex sets if and only if it absorbs all bounded sets in $M_\infty(V)$, since boundedness is preserved by taking M_∞-convex hulls.

In particular this all implies that V is completely bornological if and only if $M_\infty(V)$ is bornological.

Proposition 3.7.2
Let V be a local operator space. The following are equivalent:

i. V is completely bornological,

ii. for any W a local operator space, $\mathcal{CB}(V, W) = \mathcal{CC}(V, W)$,

iii. $V = \varinjlim\{_{\boldsymbol{p}}V : \boldsymbol{p} \in \boldsymbol{\mathfrak{B}}(V)\}$.

Proof :

This (i) $\iff$ (ii): follows immediately from Proposition 3.7.1 once we switch to the M_∞ viewpoint. We have already seen that V is completely bornological if and only if $M_\infty(V)$ is bornological. It only remains to observe that if φ is completely bounded, then φ_∞ is matrix bounded, which follows since matrix convex bounded sets are mapped to matrix convex bounded sets by φ if and only if M_∞-bounded sets are mapped to M_∞-bounded sets by φ_∞.

(ii) $\iff$ (iii): By definition $V = \varinjlim_{\boldsymbol{p}}(V)$ if and only if for any map $\varphi : V \to W$ is matrix continuous exactly when the composition of the maps

$$_{\boldsymbol{p}}V \to V \to W$$

is matrix continuous for any Minkowski matrix gauge $\boldsymbol{p}$ of a completely bounded set. But this composition is matrix continuous for such a $\boldsymbol{p}$ if and only if for any seminorm $\boldsymbol{\sigma}$ on W, there is a lambda such that $\boldsymbol{\sigma}(\varphi(v)) \leq \lambda \boldsymbol{p}(v)$, that is if and only if the image of the unit set of $\boldsymbol{p}$ is bounded. Hence $V = \varinjlim_{\boldsymbol{p}}(V)$ if and only if for any map $\varphi : V \to W$ is matrix continuous exactly when φ is completely bounded. □

3.8 Duality

Our immediate concern in this section is to consider the following problem: if we have a local operator space V and its matrix continuous dual $V' = \mathcal{CC}(V, \mathbb{C})$, for what other matrix topologies on V do we still have V' as its matrix continuous dual.

Classically this question is answered by the Arens-Mackey theorem. If V is a Hausdorff locally convex space and V' its continuous dual then a topology τ has V' as its continuous dual if and only if it has a neighborhood base at 0 consisting of the polars of a family of weakly compact convex sets in V'. The weakest of these topologies is the weak topology, the strongest is the Mackey topology consisting of the polars of all weakly compact convex sets in V'.

Let V be a Hausdorff local operator space (by which we mean that the topology on V is Hausdorff) and V' its continuous dual, then we notice that V and V' are paired as vector spaces. So for the time being we will deal with the more general situation where we simply have two paired local operator spaces V and W. If $\mathfrak{S}$ is a family of completely bounded matrix gauges of W, then we can define a matrix topology on V by using the generating seminorms $\boldsymbol{p}^{\circledcirc}$ for $\boldsymbol{p} \in \mathfrak{S}$. This follows since we know that the matrix gauges of completely bounded sets are faithful, and so their dual gauges must be matrix seminorms since $\boldsymbol{p}^{\circledcirc}(v) = \infty$ if and only if there is a $w \in W$, $w \neq 0$ such that $\boldsymbol{p}(\lambda w) \leq 1$ for all $\lambda \in [0, \infty]$, ie. $\boldsymbol{p}(w) = 0$.

The two obvious choices are to take $\mathfrak{S}$ to be all completely bounded sets, which we denote $\mathfrak{B}(V, W)$, or to be all sets containing the matrix convex hulls of single matrix points, which we denote $\mathfrak{W}(V, W)$, just as in Section 3.4. We will look at this second case first. Clearly here the matrix topology on W is irrelevant, as is the matrix topology on V, since we have been given W. So we have

Lemma 3.8.1
If V and W are two paired vector spaces and we give V the matrix topology $\mathfrak{W}(V, W)$, then $\mathcal{CC}(V, \mathbb{C}) = W$.

Proof :
Clearly every w in W is matrix continuous on V with the weak topology.

If $\varphi \in \mathcal{CC}(V, \mathbb{C})$, then there is some $w \in M_n(W)$ so that if $\boldsymbol{\rho_w}$ is the Minkowski gauge of the matrix polar of the matrix set containing just w, then

$$\|\boldsymbol{\varphi}(v)\| \leq \boldsymbol{\rho_w}(v).$$

We claim that either $\varphi = \sigma \cdot_n w$ for some $\sigma \in \mathcal{T}_n$, or there is a $v \in V$ such that $\varphi(v) = 1$, but $\langle w, v \rangle = 0$. The second alternative of the claim leads to an immediate contradiction, since clearly v is in the matrix polar of w, but $\|\boldsymbol{\varphi}(v)\| \geq \boldsymbol{\rho_w}(v)$.

Since $\varphi = \sigma \cdot_n w$ if and only if $\varphi \in \text{span}(w_{i,j})$, which is a finite dimensional subspace of W, the claim follows from a classical vector space lemma (see [33], Chapter II, Lemma 5, for example).

Now all we need to note is that $\sigma \cdot_n w \in W$. □

Corollary 3.8.2
Let V be a Hausdorff local operator space and V' its continuous dual, then V with the matrix topology $\mathfrak{W}(V, V')$ has V' as its matrix dual.

Looking more closely at what we are trying to do, we note that we are only trying to find things which agree with W as a vector space. The only matrix topology that we might expect to be of any relevance is the $\mathfrak{W}(W, V)$ matrix topology, since this is where we will hope to find our weakly matrix compact convex sets. So we will be looking at a local operator space V paired with a vector space W which we will turn into a local operator space with the $\mathfrak{W}(W, V)$ matrix topology. We also note that Lemma 3.8.1 implies that with these assumptions V and W are in weak duality if we give V the $\mathfrak{W}(V, W)$ topology, and so we can use the matricial bipolar theorem.

We firstly note that every topology of V is a matrix polar topology.

Proposition 3.8.3
Let V and W are two paired vector spaces and V is a local operator space. Then the matrix topology of V is the dual matrix topology generated by the set of matrix polars of some $\mathfrak{W}(W, V)$ closed matrix convex sets in W.

Proof :
If $\boldsymbol{X}$ is a matrix unit set for some $\boldsymbol{\rho} \in \mathfrak{N}(V)$, then $\boldsymbol{X}^{\circledcirc\circledcirc} = \boldsymbol{X}$, since if $\boldsymbol{X}$ is closed in V, then each level is weakly closed, and certainly $\boldsymbol{X}$ is matrix convex, so the result follows from the matricial bipolar theorem.

Hence the topology on V is generated by the matrix polars of the family

$$\{\boldsymbol{X}^{\circledcirc} : \boldsymbol{X} \text{ the closed unit set of } \boldsymbol{\rho} \in \mathfrak{N}(V)\}$$

of matrix subsets of W. □

We also note that the sets we are taking polars of are have certain compactness properties. If $V^{\diamond}$ is the algebraic dual of V, then we have an canonical inclusion of vector spaces $\iota : W \hookrightarrow V^{\diamond}$ given by

$$w \mapsto \langle \cdot, w \rangle$$

Moreover this inclusion is matrix continuous when we give these two spaces the appropriate weak matrix topologies, since for any $v \in M_n(V)$, $w \in M_n(W)$ lies in the matrix polar (in W) of $\boldsymbol{X_v}$ if and only if $\iota(w)$ lies in the matrix polar of $\boldsymbol{X_v}$ (in $V^{\diamond}$).

Theorem 3.8.4
Let V and W be two paired vector spaces, and V be a local operator space. If $\boldsymbol{X}$ is a matrix unit set in V, then $\boldsymbol{X}^{\circledcirc}$ is matrix $\mathfrak{W}(W, V)$-compact.

Proof :
We note immediately that $\boldsymbol{X}^{\circledcirc}$ is completely bounded in the algebraic dual $V^{\diamond}$ of V with the weak matrix topology $\mathfrak{W}(V^{\diamond}, V)$, since $\boldsymbol{X}$ is completely absorbent and so given any $v \in V$, there is a $\lambda > 0$ so that $v = \lambda x \in \lambda \boldsymbol{X}$ and so for any $\varphi \in V^{\diamond}$,

$$\boldsymbol{\rho_v}(\varphi) = ||\boldsymbol{\varphi}(v)|| = ||\boldsymbol{\varphi}(\lambda x)|| \leq \lambda.$$

Now $V^{\diamond}$ is complete in the weak matrix topology since it is complete in the weak topology, and $\boldsymbol{X}^{\circledcirc}$ is closed in the weak matrix topology. Hence $\boldsymbol{X}^{\circledcirc}$ is closed and weakly bounded at each level, and so is compact at each level. So $\boldsymbol{X}^{\circledcirc}$ is weakly compact in $V^{\diamond}$. Hence $\boldsymbol{X}^{\circledcirc}$ is weakly compact in W, since the inverse images of these compact sets under ι is compact. □

Corollary 3.8.5
If V is a Hausdorff local operator space, then the matrix polar of the matrix unit ball of any continuous seminorm is matrix compact in V' with the weak matrix topology.

Corollary 3.8.6
If V is an operator space, then the dual matrix unit ball is weakly matrix compact.

This last corollary is obviously an operator space version of Alaoglu's theorem, and once you unravel the definitions, it was observed by Weaver [40].

So at this point we have proved half of the Arens-Mackey result for local operator spaces—we have shown that the matrix topology on V is generated by the polars of weakly matrix compact convex sets in the dual. We want to show that every such matrix topology gives you the dual back

Theorem 3.8.7
Let V and W be paired vector spaces and V be a local operator space. Then W is the continuous dual of V for some matrix topology if and only if this matrix topology is generated by the matrix polars of a family of $\mathfrak{W}(W,V)$ matrix compact convex sets in W.

Proof :

We observed in Proposition 3.8.3 that the new matrix topology of V is the topology generated by the matrix polars of matrix unit balls, and that these matrix polars are matrix $\mathfrak{W}(W,V)$-compact convex sets in W by Theorem 3.8.4.

Conversely if the new matrix topology on V is the dual topology generated by a family of $\mathfrak{W}(V)$-compact convex sets in W, then the continuous dual V' of the new V sits inside $V^\diamond$, however $\varphi \in V^\diamond$ is matrix continuous if and only if it lies in the matrix polar of a matrix unit ball of V, but if we embed W in $V^\diamond$, then by the matrix bipolar theorem, every such element lies in the image of W and every element of W is such an element. So $V' = W$ as vector spaces. □

We will call the topology of on V generated by all the weakly matrix compact convex subsets of V' the Mackey matrix topology.

3.9 Operator Nuclearity

In classical real and functional analysis, many important and useful locally convex spaces come from a class known as nuclear spaces. These include the Schwartz spaces, spaces of tempered distributions and $C^\infty(M)$ for M a compact manifold.

Given Banach spaces V and W, we say that a linear $\varphi : V \to W$ is *nuclear* if it lies in the image of the canonical map

$$\Phi : V^* \hat{\otimes} W \to V^* \check{\otimes} W \cong \mathcal{B}(V,W).$$

The space $\mathcal{N}(V,W)$ of all nuclear maps from V to W is the image of this map with the quotient norm $\|\cdot\|_{\mathrm{nuc}}$ coming from

$$\mathcal{N}(V,W) \cong \frac{V^* \check{\otimes} W}{\ker \Phi}.$$

A map φ is in $\mathcal{N}(V,W)$ with $\|\varphi\|_{\mathrm{nuc}}$ if and only if it factors through a commutative diagram

$$\begin{array}{ccc} \ell_\infty & \xrightarrow{\theta_\lambda} & \ell_1 \\ {\scriptstyle\sigma}\uparrow & & \downarrow{\scriptstyle\tau} \\ V & \xrightarrow{\varphi} & W \end{array}$$

where the vertical maps σ and τ are contractions, and θ_λ is multiplication by some $\lambda \in \ell_1$ with $\|\lambda\| \leq 1$. We can in fact replace ℓ^∞ by c_0 in the above definition, if needs be.

We say that a locally convex space V is *nuclear* if for every $\rho \in \mathfrak{N}(V)$ there is a $\sigma \in \mathfrak{N}(V)$ with $\sigma \geq \rho$ so that the connecting map $\pi_{\rho,\sigma} : \bar{V}_\sigma \to \bar{V}_\rho$ is nuclear.

Theorem 3.9.1
Let V be a locally convex space. Then the following are equivalent:

i. V is nuclear,

ii. for all locally convex spaces W we have that $V \underset{\frown}{\otimes} W = V \underset{\smile}{\otimes} W$.

The analogue of nuclear maps for operator spaces were studied by Effros and Ruan in [13]. We say that a map is *operator* or *matrix nuclear* if it lies in the range of the map

$$\mathbf{\Phi} : V^* \widehat{\otimes}_{\text{op}} W \to V^* \check{\otimes}_{\text{op}} W \subseteq \mathcal{CB}(V, W).$$

We let $\mathcal{CN}(V, W)$ be this image with the quotient matrix norm $\|\cdot\|_{\text{opnuc}}$ from

$$\mathcal{CN}(V, W) \cong \frac{V^* \widehat{\otimes}_{\text{op}} W}{\ker \mathbf{\Phi}}.$$

We have that $\varphi \in \mathcal{CN}(V, W)$ with operator nuclear norm < 1 if and only if we can factor φ through a commutative diagram

$$\begin{array}{ccc} \mathcal{B} & \xrightarrow{\theta_{a,b}} & \mathcal{T} \\ {\scriptstyle\sigma}\uparrow & & \downarrow{\scriptstyle\tau} \\ V & \xrightarrow[\varphi]{} & W \end{array}$$

where σ and τ are complete contractions and $\theta_{a,b}$ is the map

$$\theta_{a,b}(x) = axb$$

where a and $b \in \mathcal{HS}$ with Hilbert-Schmidt norm strictly less than 1. In fact we can replace $\mathcal{B}$ with $\mathcal{K}$ in the above factorization, to get

$$\begin{array}{ccc} \mathcal{K} & \xrightarrow{\theta_{a,b}} & \mathcal{T} \\ {\scriptstyle\sigma}\uparrow & & \downarrow{\scriptstyle\tau} \\ V & \xrightarrow[\varphi]{} & W \end{array} \tag{3.9}$$

and this fact will be important in our analysis, as $\mathcal{K}$ is much better behaved with respect to tensor products.

The equivalence of these notions is seen by careful inspection of the operator norms. Given $u \in M_p(V^* \widehat{\otimes} W)$ with $\|u\|_{\wedge,p} \leq 1$, we can write it as $u = \alpha(\xi \otimes w)\beta$ for some $\xi = [\xi_{i,j}] \in \mathcal{K}(V^*)$, $w = [w_{k,l}] \in \mathcal{K}(W)$, and $\alpha = [\alpha_{\lambda,(i,k)}] \in \mathcal{B}(\ell^\infty \otimes \ell^\infty, \mathbb{C}^p)$, $\beta = [\beta_{(j,l),\mu}] \in \mathcal{B}(\mathbb{C}^p, \ell^\infty \otimes \ell^\infty)$ each with norm < 1 (Proposition 3.1 of [9]). That is

$$u = \left[\sum_{i,j,k,l} \alpha_{\lambda,(i,k)} \xi_{i,j} \otimes w_{k,l} \beta_{(j,l),\mu} \right].$$

We can regard $\mathcal{B}(\ell^\infty \otimes \ell^\infty, \mathbb{C}^p)$ as $M_{p,1}(\mathcal{HS}^*)$ and $\mathcal{B}(\mathbb{C}^p, \ell^\infty \otimes \ell^\infty)$ as $M_{1,p}(\mathcal{HS})$. Letting $a \in M_{p,1}(\mathcal{HS})$ be such that $a_{\lambda,1} = [\alpha_{\lambda,(k,i)}]$, and $b = \beta$ we have

$$u = \sum_{k,l} (a\xi b)_{k,l} \otimes w_{k,l}.$$

Now if $\varphi = \mathbf{\Phi}_p(u) \in M_p(\mathcal{CB}(V^*, W))$, then

$$\varphi(v) = \sum_{k,l} (a\xi(v)b)_{k,l} \otimes w_{k,l}$$

and noting that $\theta_{a,b}(\xi(v)) \in M_p(\mathcal{T})$, we have

$$\varphi = \sum_{k,l} (\theta_{a,b} \circ \xi)_{k,l} \otimes w_{k,l}.$$

But as operator spaces we have complete isometries $\mathcal{K}(V^*) \to \mathcal{CB}(V, \mathcal{K})$ and $\mathcal{K}(W) \to \mathcal{CB}(\mathcal{T}, W)$, so if we let σ be the complete contraction corresponding to ξ and τ be the complete contraction corresponding to w, we have

$$\varphi = \tau_p \circ \theta_{a,b} \circ \sigma.$$

So $\|\varphi\|_{\mathcal{CN},p} < 1$ implies that φ factors as in (3.9).

If φ factors through $\mathcal{K}$, then it trivially factors through $\mathcal{B}$. Completing the third direction, if φ factors through $\mathcal{B}$ and $\mathcal{T}$ then we have a complete isometry $\mathcal{CB}(\mathcal{T}, W) \to \mathcal{B}(W)$, where $\mathcal{B}(W)$ is the space of infinite matrices $w = [w_{i,j}]$ with entries $w_{i,j} \in W$ and $\sup \|w_{i,j}\| < \infty$. The norm on $\mathcal{B}(W)$ is

$$\|w\| = \sup \|w_{i,j}\|.$$

Similarly, we have a complete isometry $\mathcal{CB}(V^*, \mathcal{B}) \to \mathcal{B}(V^*)$, and reversing the above argument, we can write

$$\varphi = \mathbf{\Phi}_p(u)$$

where

$$u = \alpha(\xi \otimes w)\beta$$

for some $\xi = [\xi_{i,j}] \in \mathcal{K}(V^*)$, $w = [w_{k,l}] \in \mathcal{K}(W)$, and $\alpha = [\alpha_{\lambda,(i,k)}] \in \mathcal{B}(\ell^\infty \otimes \ell^\infty, \mathbb{C}^p)$, $\beta = [\beta_{(j,l),\mu}] \in \mathcal{B}(\mathbb{C}^p, \ell^\infty \otimes \ell^\infty)$ each with norm < 1. But by Proposition 3.1 of [9], this implies $\|u\|_{\wedge,p} < 1$.

Since we have that $\widehat{\otimes}_{\text{op}}$ and $\check{\otimes}_{\text{op}}$ define cross norms on $V^* \otimes W$, we have the following diagram

$$\begin{array}{ccccccc}
V^* \widehat{\otimes} W & \longrightarrow & V^* \widehat{\otimes}_{\text{op}} W & \longrightarrow & V^* \check{\otimes}_{\text{op}} W & \longrightarrow & V^* \check{\otimes} W \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\mathcal{N}(V,W) & \longrightarrow & \mathcal{CN}(V,W) & \longrightarrow & \mathcal{CB}(V,W) & \longrightarrow & \mathcal{B}(V,W)
\end{array}$$

and so we see that if $\varphi : V \to W$ is nuclear it is automatically operator nuclear. This fact can be seen much more concretely by simply embedding ℓ^∞ and ℓ^1 along the diagonals of $\mathcal{B}$ and $\mathcal{T}$ respectively, and letting a and b be the operators taking the standard basis element e_k to $\sqrt{\lambda_k} e_k$.

We will need the following two propositions about tensor products and operator nuclear maps when we look at tensor products of the analogues of nuclear locally convex spaces.

Proposition 3.9.2
Let V and W be operator spaces. If $\varphi : V \to W$ is an operator nuclear mapping with $\|\varphi\|_{\mathcal{CN}} < 1$, then for any operator space X, the map

$$\varphi \otimes \mathrm{id} : V \otimes_{op} X \to W \otimes_{op} X$$

is a complete contraction, and so extends to a complete contraction

$$\varphi \otimes \mathrm{id} : V \check{\otimes}_{op} X \to W \check{\otimes}_{op} X.$$

Proposition 3.9.3
If V, W, X, and Y are operator spaces, and $\varphi : V \to W$ and $\psi : X \to Y$ are operator nuclear maps, then

$$\varphi \otimes \psi : V \widehat{\otimes}_{op} X \to W \widehat{\otimes}_{op} Y$$

is operator nuclear.

Proof :

Assume that we have φ factoring as in (3.9), and ψ factoring as

(3.10)
$$\begin{array}{ccc}
\mathcal{K} & \xrightarrow{\theta_{c,d}} & \mathcal{T} \\
{\scriptstyle\rho}\uparrow & & \downarrow{\scriptstyle\gamma} \\
X & \xrightarrow[\varphi]{} & Y
\end{array}$$

Then by the functoriality of $\widehat{\otimes}_{\mathrm{op}}$ we have

$$\begin{array}{ccc}
\mathcal{K}\,\check{\otimes}_{\mathrm{op}}\,\mathcal{K} \;\cong\; \mathcal{K}(\ell^2 \otimes \ell^2) & \xrightarrow{\theta_{a\otimes c,b\otimes d}} & \mathcal{T}(\ell^2 \otimes \ell^2) \;\cong\; \mathcal{T}\,\widehat{\otimes}_{\mathrm{op}}\,\mathcal{T} \\
{\scriptstyle\sigma\otimes\rho}\uparrow & & \downarrow{\scriptstyle\tau\otimes\gamma} \\
V\,\widehat{\otimes}_{\mathrm{op}}\,X & \xrightarrow[\varphi\otimes\psi]{} & W\,\widehat{\otimes}_{\mathrm{op}}\,Y
\end{array}$$

where $\sigma \otimes \rho$ and $\tau \otimes \gamma$ are complete contractions and $a \otimes c$ and $b \otimes d$ are in $\mathcal{HS}(\ell^2 \otimes \ell^2)$. It only remains to note that $\ell^2 \otimes \ell^2 \cong \ell^2$ as Hilbert spaces, so $\varphi \otimes \psi$ is operator nuclear. $\square$

We are now ready to define exactly what we mean by a matrix nuclear local operator space, although this definition should come as no surprise.

Definition 3.9.1
We say that a local operator space is matrix nuclear *if for any continuous operator seminorm $\boldsymbol{\rho} \in \mathfrak{N}(V)$, there is a $\boldsymbol{\sigma} \in \mathfrak{N}(V)$ with $\boldsymbol{\sigma} \geq \boldsymbol{\rho}$ such that the connecting map*

$$\pi_{\boldsymbol{\rho},\boldsymbol{\sigma}} : \bar{V}_{\boldsymbol{\sigma}} \to \bar{V}_{\boldsymbol{\rho}}$$

is operator nuclear.

Theorem 3.9.4
Let V be a local operator space. Then V is a matrix nuclear local operator space if and only if it is a nuclear locally convex space.

Proof :

If V is a nuclear locally convex space, then given $\boldsymbol{\rho} \in \mathfrak{N}(V)$, we can find a $\sigma \in \mathfrak{N}(V)$ such that the Banach space map $\bar{V}_\sigma \to \bar{V}_{\rho_1}$ is nuclear. But since the matrix seminorms on V are assumed to generate the locally convex topology of V, we can find $\boldsymbol{\tau} \in \mathfrak{N}(V)$ so that $\tau_1 \geq \sigma$ and so the composition map $\bar{V}_{\boldsymbol{\tau}} \to \bar{V}_{\boldsymbol{\rho}}$ is a nuclear map between operator spaces. Hence it is operator nuclear, and so V is a matrix nuclear local operator space.

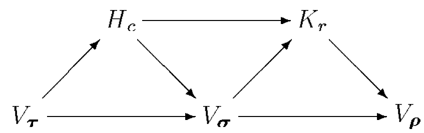
Conversely, if V is a matrix nuclear local operator space, then for any $\boldsymbol{\rho} \in \mathfrak{N}(V)$ we can find $\boldsymbol{\sigma}$ and $\boldsymbol{\tau} \in \mathfrak{N}(V)$ with $\boldsymbol{\tau} \geq \boldsymbol{\sigma} \geq \boldsymbol{\rho}$ so that the maps

$$\bar{V}_\sigma \to \bar{V}_\rho$$

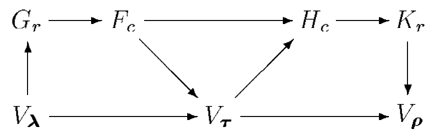
and

$$\bar{V}_\tau \to \bar{V}_\sigma$$

are operator nuclear. But any operator nuclear map can be factored through both a Hilbert row space and a Hilbert column space, so we get the following diagram of complete contractions:



where H and K are Hilbert spaces, and the top row is simply the composition of the maps $H_c \to \bar{V}_\sigma \to K_r$. However a completely contractive map from $H_c \to K_r$ lies in $\mathcal{HS}(H, K)$ (see Effros and Ruan [10], Corollary 4.5). We can find a $\boldsymbol{\lambda} \in \mathfrak{N}(V)$ $\boldsymbol{\lambda} \geq \boldsymbol{\tau}$ so that the connecting maps factor like so



where again the map $G_r \to H_c$ is just the composition of the maps through $\bar{V}_\tau$. Since the chain of maps from F though to K contains two Hilbert-Schmidt maps, the composition must be trace class. But $\mathcal{T}(F, K)$ are exactly the nuclear maps from F to K, and since the maps $V_\lambda \to F$ and $K \to V_\rho$ are contractions, the factorization we have for the trace class map through ℓ^∞ and ℓ^1 extends to a factorization for the connecting map from $\bar{V}_\lambda$ to $\bar{V}_\rho$. Hence V is a nuclear locally convex space. □

This result might appear problematical, if it were not for the following surprising fact.

Theorem 3.9.5
If V is a nuclear locally convex space, then it has precisely one quantization.

Proof :
We need only show that $\min V = \max V$. We observe that, since V is nuclear $M_\infty \underset{\vee}{\otimes} V = M_\infty \underset{\wedge}{\otimes} V$. Also we have that the identity map $M_\infty(\max V) \to M_\infty(\min V)$ is continuous, since the map of (3.1) is matrix continuous. Finally (3.7) and (3.8) allow us to factor the identity map as a sequence of continuous maps

$$M_\infty(\min V) = M_\infty \underset{\vee}{\otimes} V = M_\infty \underset{\wedge}{\otimes} V \to M_\infty(\max V) \to M_\infty(\min V).$$

Hence $M_\infty(\min V) = M_\infty(\max V)$ as locally convex spaces, and so $\min V = \max V$ as local operator spaces.
□

The equivalent statement in the operator space case (up to isomorphism) is true only for finite dimensional operator spaces. Indeed, finite dimensional spaces are the only nuclear Banach spaces, and so are the only matrix nuclear operator spaces.

An obvious question is whether there is a result parallel to Theorem 3.9.1. One direction of the classical equivalence is easy to show for local operator spaces.

Theorem 3.9.6
If V is a matrix nuclear local operator space, then for any local operator space W we have that

$$V \underset{\wedge}{\otimes}_{op} W = V \underset{\vee}{\otimes}_{op} W.$$

Proof :
We know that the extension $\boldsymbol{\Phi}$ of the identity map is matrix continuous from $V \underset{\wedge}{\otimes}_{\rm op} W$ to $V \underset{\vee}{\otimes}_{\rm op} W$. In the other direction, we have that for any $\boldsymbol{\rho} \in \mathfrak{N}(V)$ and $\boldsymbol{\sigma} \in \mathfrak{N}(W)$, we can find a $(\tau) \in \mathfrak{N}(V)$ with $\boldsymbol{\tau} \geq \boldsymbol{\rho}$ and the connecting map operator nuclear. So by Proposition 3.9.3, the we have that the identity map induces a complete contraction

$$\bar{V}_{\boldsymbol{\tau}} \check{\otimes}_{\rm op} \bar{W}_{\boldsymbol{\sigma}} \to \bar{V}_{\boldsymbol{\rho}} \hat{\otimes}_{\rm op} \bar{W}_{\boldsymbol{\sigma}}$$

and so the identity map is matrix continuous from

$$V \underset{\smile}{\otimes}_{\mathrm{op}} W \to V \underset{\frown}{\otimes}_{\mathrm{op}} W.$$

□

The other direction is more problematical, and we need to introduce some new ideas to explain what is going on. One of the key differences between the theory of operator spaces and the theory of Banach spaces, is that while all Banach spaces are locally reflexive the analogous condition is not always true for operator spaces. More precisely, a Banach space V is *locally reflexive* if for each finite dimensional Banach space W we have that any contraction $W \to V^{**}$ can be approximated in the point-norm topology by contractions $W \to V$.

For an operator space V, we say that it is *locally operator reflexive* if for every finite dimensional operator space W, any complete contraction $W \to V^{**}$ can be approximated in the point-norm topology by complete contractions $W \to V$. As we will see, this is the missing ingredient that will allow us to formulate a version of Theorem 3.9.1. Effros and Ruan showed in [13], Theorem 3.6, that local operator reflexivity of V is equivalent to the condition that for all operator spaces W,

$$\mathcal{CI}(W, V^*) \to (V \widehat{\otimes}_{\mathrm{op}} W)^*$$

is a completely isometric bijection, where $\mathcal{CI}(X, Y)$ are the operator integral maps from X to Y. A map $\varphi : X \to Y$ is *operator integral* if there is a diagram

$$\begin{array}{ccc} \mathcal{B} & \xrightarrow{\theta_{a,b}} & \mathcal{T} \\ {\scriptstyle\sigma}\uparrow & & \downarrow{\scriptstyle\tau} \\ X & \xrightarrow[\varphi]{} & Y \end{array}$$

which approximately commutes (in the point-norm topology). As one might expect, this is the operator version of a classical family of maps. The similarity between this diagram and (3.9) should hint that the two notions are connected: clearly all operator nuclear maps are operator integral; what is less clear is that the composition of two integral maps is nuclear. This second claim stems from the fact that operator integral maps are also Hilbert row and column factorable, and that all completely bounded maps from H_c to H_r are operator nuclear (see [12]).

In particular, this means that a local operator space is matrix nuclear if and only if for every continuous matrix seminorm $\boldsymbol{\rho}$, there is a $\boldsymbol{\sigma} \geq \boldsymbol{\rho}$ such that the connecting map is operator integral (indeed, we could replace operator integral here by absolute matrix summing, or simply that the map be both Hilbert row and column factorable).

Theorem 3.9.7
A local operator space V is matrix nuclear if and only if we have

i. *for some generating family of continuous matrix seminorms $\mathfrak{R}$, the operator spaces $\bar{V}^*_{\boldsymbol{\rho}}$ are locally reflexive for all $\boldsymbol{\rho} \in \mathfrak{R}$,*

ii. *$V \underset{\frown}{\otimes}_{op} W = V \underset{\smile}{\otimes}_{op} W$ for all operator spaces W.*

Proof :

We have already shown that a matrix nuclear space satisfies (ii). V satisfies (i), since it can be written as a projective limit of Hilbert row spaces: given any continuous matrix seminorm $\boldsymbol{\rho}$, we find $\boldsymbol{\sigma} \geq \boldsymbol{\rho}$ so that $\pi_{\boldsymbol{\rho},\boldsymbol{\sigma}}$ is nuclear. Then we can factor $\pi_{\boldsymbol{\rho},\boldsymbol{\sigma}} = \tau \circ \gamma$ through a Hilbert row space H_r, and get a continuous matrix seminorm

$$\tilde{\boldsymbol{\rho}}(v) = \|\gamma(\pi_{\boldsymbol{\sigma}}(v))\|.$$

The set $\mathfrak{R} = \{\tilde{\rho}\}$ is a generating family for the matrix topology of V. All that is required is to observe that Hilbert row spaces are reflexive, and hence locally reflexive.

Conversely, suppose we are given $\boldsymbol{\rho} \in \mathfrak{N}(V)$. Then the map $\pi_{\boldsymbol{\rho}} : V \to \bar{V}_{\boldsymbol{\rho}}$ gives us an element $\varphi \in (V \underset{\smile}{\otimes}_{\mathrm{op}} \bar{V}_{\boldsymbol{\rho}}^*)'$ by

$$\varphi(v \otimes \phi) = \phi(\pi_{\boldsymbol{\rho}}(v)).$$

(ii) tells us that in fact $\varphi \in (V \otimes_{\mathrm{op}} \bar{V}_{\boldsymbol{\rho}}^*)'$. Now we have that

$$V \otimes_{\mathrm{op}} \bar{V}_{\boldsymbol{\rho}}^* = \varprojlim \bar{V}_{\boldsymbol{\sigma}} \breve{\otimes}_{\mathrm{op}} \bar{V}_{\boldsymbol{\rho}}$$

and so there is some $\boldsymbol{\sigma}$ such that the induced map

$$\bar{\varphi} : \bar{V}_{\boldsymbol{\sigma}} \breve{\otimes}_{\mathrm{op}} \bar{V}_{\boldsymbol{\rho}}^* \to \mathbb{C}$$

such that

$$\varphi(v \otimes \phi) = \bar{\varphi}(\pi_{\boldsymbol{\rho}}(v) \otimes \phi)$$

is completely bounded, ie. $\bar{\varphi} \in (\bar{V}_{\boldsymbol{\sigma}} \breve{\otimes}_{\mathrm{op}} \bar{V}_{\boldsymbol{\rho}}^*)^*$. By choosing $\boldsymbol{\sigma}$ sufficiently large, we can assume that $\boldsymbol{\sigma} \geq \boldsymbol{\rho}$ and that $V_{\boldsymbol{\sigma}}^*$ is locally reflexive, by (i). That is we can assume that

$$(\bar{V}_{\boldsymbol{\sigma}} \breve{\otimes}_{\mathrm{op}} \bar{V}_{\boldsymbol{\rho}}^*)^* = \mathcal{CI}(V_{\boldsymbol{\sigma}}, V_{\boldsymbol{\rho}}^{**}).$$

So now we have that

$$\begin{aligned} \bar{\varphi}(\pi_{\boldsymbol{\sigma}}(v) \otimes \psi) &= \varphi(v \otimes \psi) \\ &= \psi(\pi_{\boldsymbol{\rho}}(v)) \\ &= \psi(\pi_{\boldsymbol{\sigma},\boldsymbol{\rho}}(\pi_{\boldsymbol{\sigma}}(v))). \end{aligned}$$

So $\pi_{\boldsymbol{\sigma},\boldsymbol{\rho}}$ is exactly $\bar{\phi}$ regarded as a map from $\bar{V}_{\boldsymbol{\sigma}}$ to $\bar{\boldsymbol{\rho}}^{**}$, so $\pi_{\boldsymbol{\sigma},\boldsymbol{\rho}} \in \mathcal{CI}(V_{\boldsymbol{\sigma}}, V_{\boldsymbol{\rho}}^{**})$. But in fact $\pi_{\boldsymbol{\sigma},\boldsymbol{\rho}}$ maps into $\bar{V}_{\boldsymbol{\rho}} \subseteq \bar{V}_{\boldsymbol{\rho}}^{**}$, however we have a natural inclusion

$$\mathcal{CI}(V_{\boldsymbol{\sigma}}, V_{\boldsymbol{\rho}}) \hookrightarrow \mathcal{CI}(V_{\boldsymbol{\sigma}}, V_{\boldsymbol{\rho}}^{**})$$

and it is a complete isometry ([13], Proposition 3.3). So we have shown that for any $\boldsymbol{\rho}$ we can find an $\boldsymbol{\sigma} \geq \boldsymbol{\rho}$ such that the connecting map is operator integral. Hence by the preamble to the statement of the theorem V is in fact operator nuclear. □

We conclude this section with another result on tensor products.

Proposition 3.9.8
If V and W are both matrix nuclear local operator spaces, then so is $V \underset{\smile}{\otimes}_{op} W$.

Proof :
This follows immediately from Proposition 3.9.3: given $\boldsymbol{\rho} \in \mathfrak{N}(V)$ and $\boldsymbol{\sigma} \in \mathfrak{N}(W)$ we find $\boldsymbol{\rho}' \geq \boldsymbol{\rho}$ and $\boldsymbol{\sigma}' \geq \boldsymbol{\sigma}$ so that the respective connecting maps are matrix nuclear, but then the map

$$\pi_{\boldsymbol{\rho},\boldsymbol{\rho}'} \otimes \pi_{\boldsymbol{\sigma},\boldsymbol{\sigma}'} : \bar{V}_{\boldsymbol{\rho}'} \widehat{\otimes}_{\mathrm{op}} \bar{W}_{\boldsymbol{\sigma}'} \to \bar{V}_{\boldsymbol{\rho}} \widehat{\otimes}_{\mathrm{op}} \bar{W}_{\boldsymbol{\sigma}}$$

is a matrix nuclear. □

This of course implies that $V \otimes_{\mathrm{op}} W$ and $V \otimes_h W$ are also operator nuclear, since they are both completely isomorphic to $V \underset{\smile}{\otimes}_{\mathrm{op}} W$.

Example 3.9.1
We note that Theorem 3.9.4 implies that the local operator spaces of Examples 3.1.4 and 3.2.3 are all matrix nuclear, and Theorem 3.9.5 implies that they are all the same.

Example 3.9.2
The spaces W_c and W_r of Example 3.1.6 are nuclear, and hence matrix nuclear. Indeed, they are completely isomorphic. Proposition 3.9.8 implied that the space T of physical matrix state space is also matrix nuclear, and must be the unique quantization of the physical state space. The algebra of observables $\mathcal{A}$ is also nuclear [6], and since the local operator structure we have constructed over it is a quantization of its usual topology, we have that our version is matrix nuclear.

3.10 The Kernel Theorem

The classical kernel theorem is a result which tells you what bilinear maps on countably generated nuclear spaces look like. Since most nuclear spaces of classical interest, such as the Schwartz spaces and the various spaces of distributions, are countably generated, this result has particular applications in allowing one to perform calculations with these bilinear maps.

The first step to proving the kernel theorem is a lemma which is a corollary of the uniform complete boundedness principle. We say that a bilinear form $\varphi : V \times W \to X$ is *separately matrix continuous* if for every n and each $v \in M_n(V)$, $\varphi_{n;1}(v, \cdot) : W \to M_n(X)$ is matrix continuous, and for each $w \in M_n(W)$, $\varphi_{1;n}(\cdot, w) : V \to M_n(X)$ is matrix continuous.

Lemma 3.10.1
Let V and W be countably generated local operator spaces, X an operator space. Let $\varphi : V \times W \to X$ a separately matrix continuous bilinear form. If W is matrix barrelled, then φ is (jointly) matrix continuous.

Proof :
We firstly note that φ is matrix continuous if and only if

$$\varphi_{\infty;\infty} : M_\infty(V) \times M_\infty(W) \to M_{\infty\times\infty}(X)$$

is continuous. Moreover since the matrix topologies are countably generated, the topologies on $M_\infty(V)$ and $M_\infty(W)$ are countably generated, and so $\varphi_{\infty;\infty}$ is continuous if and only if given any sequences $v_n \to 0$ and $w_n \to 0$ in $M_\infty(V)$ and $M_\infty(W)$ respectively, we have that

$$\varphi_{\infty;\infty}(v_n, w_n) \to 0$$

as $n \to \infty$. In other words, φ is matrix continuous if and only if

$$||\varphi_{r_n;s_n}(v_n, w_n)|| \to 0$$

as $n \to \infty$.

Now for each v_n, let $\psi_n(w) = \varphi_{r_n}(v_n, w) \in M_{r_n}(\mathcal{CC}(W, \mathbb{C}))$. Now since $x_n \to 0$ and $\boldsymbol{\varphi}(\cdot, w) \in \mathcal{CC}(V, M_n)$ we have that the matrix set with levels

$$X_k(w) = \{\boldsymbol{\psi_n}(w) : r_n = k\}$$

is completely bounded for every $w \in M_l(W)$, and $l \in \mathbb{N}$. Hence the matrix set of maps whose elements are the ψ_n is pointwise completely bounded. Now since W is barrelled, this matrix set is also matrix equicontinuous, that is, given $\boldsymbol{\gamma} \in \mathfrak{N}(V)$, we can find a matrix seminorm $\boldsymbol{\rho} \in \mathfrak{N}(W)$ such that

$$\boldsymbol{\gamma}(\boldsymbol{\psi_n}(w)) \leq \boldsymbol{\rho}(w)$$

for all w.

But then

$$\boldsymbol{\gamma}(\boldsymbol{\varphi}(v_n, w_n)) = \boldsymbol{\gamma}(\boldsymbol{\psi_n}(w_n)) \leq \boldsymbol{\rho}(w_n) \to 0$$

as $n \to \infty$. □

The proof of the kernel theorem for local operator spaces is now easy, as we have done most of the hard work.

Theorem 3.10.2 (Kernel Theorem)
Let V be a matrix Frechet nuclear space, W be a countably generated local operator space, and X a local operator space. If $\varphi : V \times W \to X$ is separately matrix continuous in each argument, then

$$\boldsymbol{\varphi}(v, w) = \boldsymbol{\psi}(v \otimes w).$$

for some $\psi \in \mathcal{CC}(V \hat{\otimes}_{op} W, X) \cong \mathcal{CC}(V \check{\otimes}_{op} W, X)$.

Proof :

We first note that V is matrix barrelled, since it is the maximal quantization of a Frechet space, and Frechet spaces are barrelled.

This implies by Lemma 3.10.1 that φ is matrix continuous. But we also know that by Lemma 3.3.3, the space of matrix continuous bilinear forms is isomorphic to $\mathcal{CC}(V \hat{\otimes}_{\rm op} W, X)$ and so we are done. $\square$

We note that we could weaken the hypotheses to having V matrix barrelled and matrix nuclear, however we state it this way to highlight the similarities with classical formulations. The converse is somewhat trivially true due to Lemma 3.3.3, it holds for general spaces and implies joint matrix continuity.

Corollary 3.10.3
Let $\varphi : \mathcal{S}(\mathbb{R}^n) \times \mathcal{S}(\mathbb{R}^m) \to \mathbb{C}$ be separately matrix continuous. Then there is a unique $\psi \in \mathcal{S}(\mathbb{R}^{n+m})'$ such that

$$\boldsymbol{\varphi}(v, w) = \boldsymbol{\psi}(v \otimes w).$$

Corollary 3.10.4
Let $\varphi : C_c^\infty(\mathbb{R}^n) \times C_c^\infty(\mathbb{R}^m) \to \mathbb{C}$ be separately matrix continuous. Then there is a unique $\psi \in C_c^\infty(\mathbb{R}^{n+m})'$ such that

$$\boldsymbol{\varphi}(v, w) = \boldsymbol{\psi}(v \otimes w).$$

Chapter 4

Analogues of Compactness in Operator Spaces

In this chapter we are concerned with exactly what it means for a matrix set to be "compact". There are a number of ways that compact sets can be characterized in the classical theory of Banach spaces, the most familiar of these being that compact sets are the closed and totally bounded sets. They can also be characterized as being closed subsets of the closed convex hulls of sequences converging to zero. We will show that these two notions are distinct in the theory of operator spaces, and that the reasons are very deep. In particular they are intimately connected with the approximation properties for operator spaces and the relationship between these and exactness.

This line of research developed out of the search for an appropriate definition of a compact set for use in the Arens-Mackey duality theory of Section 3.8. The above two notions appear to be special cases of the definition of compactness that is used in that section and in Webster and Winkler [41], a definition which seems much better adapted to notions of duality.

Perhaps even more so than the previous chapters, the bimodule approach is critical in the theory presented in this chapter, allowing us to use classical techniques in the operator setting. The primary difference is that since we are concentrating on operator spaces, we can use completions more freely, and so we use $\mathcal{K}$-bimodules and -convexity rather than M_∞ as in the previous section.

4.1 Definitions of Matrix Compactness

If V is a Banach space, compact sets can be characterized as being closed subsets of the closed convex hulls of sequences which converge to zero in V (see [24], Proposition 1.e.2). Without loss of generality, we can take the absolutely convex hulls, instead of convex hulls. We can think of sequences which converge to 0 as being elements of $c_0(V) = c_0 \check{\otimes} V$, and given an $x \in c_0 \check{\otimes} V$, we can define

$$\operatorname{co} x = \{v \in V : v = (\tau \otimes \mathrm{id})(x), \tau \in c_c, \|\tau\|_1 \leq 1\}.$$

So we are saying that K is compact if there is some $x \in c_0 \check{\otimes} V$ such that

$$K \subseteq \overline{\operatorname{co} x}.$$

We have seen that by substituting $\mathcal{K}$ for c_0 at strategic places in a definition for Banach spaces, we can often find the correct definition of a concept for operator spaces. Let V be an operator space and let $x \in \mathcal{K}(V) \cong \mathcal{K} \otimes_{\mathrm{op}} V$, then we define the *absolutely matrix convex hull* of x to be $\mathbf{co}\, x$ where

$$\mathbf{co}\, x_k = \{v \in M_k(V) : v = (\sigma \otimes \mathrm{id})(x), \sigma \in M_k(M_\infty), \|\sigma\|_{\mathcal{T}} \leq 1\}.$$

We note that this is an absolutely convex set, since if $v \in \mathbf{co}\, x_k$, $w \in \mathbf{co}\, x_k$, then $v = (\sigma \otimes \mathrm{id})(x)$, $w = (\tau \otimes \mathrm{id})(x)$ and

$$v \oplus w = (\sigma \oplus \tau \otimes \mathrm{id})(x)$$

where $\sigma \oplus \tau \in M_{k+l}(M_\infty)$ and the unit ball of $\mathcal{T}$ is absolutely convex. Similarly

$$\alpha v \beta = (\alpha \sigma \beta \otimes \mathrm{id})(x)$$

and $\alpha\sigma\beta \in M_l(M_\infty)$ and again, the unit ball of $\mathcal{T}$ is absolutely convex.

We also note that if $\sigma \in M_k(\mathcal{K})$, then $(\sigma \otimes \mathrm{id})(x) \in \overline{\mathbf{co}(x)}$, since if we let $\sigma_n(v) = \sigma(p_n v p_n)$, then $p_n v p_n \to v$ as $n \to \infty$ and so $(\sigma_n \otimes \mathrm{id})(x) \to (\sigma \otimes \mathrm{id})(x)$ as $n \to \infty$.

Definition 4.1.1
If $\boldsymbol{K}$ is a matrix subset of an operator space V, then we say that $\boldsymbol{K}$ is operator compact *if $\boldsymbol{K}$ is closed and there is some $x \in \mathcal{K}(V)$ such that $\boldsymbol{K} \subseteq \overline{\mathbf{co}\, x}$.*

Example 4.1.1
Consider the matrix unit ball of $\mathcal{T}_n$. Let $\tau \in M_n(\mathcal{T}_n) \cong \mathcal{CB}(M_n, M_n)$ be the identity map, or equivalently

$$\tau = \begin{bmatrix} \tau_{1,1} & \cdots & \tau_{1,n} \\ \vdots & \ddots & \vdots \\ \tau_{n,1} & \cdots & \tau_{n,n} \end{bmatrix} = \sum_{i,j=1}^{n} e_{i,j} \otimes \tau_{i,j}$$

where $\tau_{i,j}(a) = a_{i,j}$. Then given any σ in the matrix unit ball of $\mathcal{T}_n$, we have that

$$\sigma(a) = \sigma(\tau(a)) = (\sigma \otimes \mathrm{id})(\sum_{i,j=1}^{n} e_{i,j} \otimes \tau_{i,j})(a)$$

and so $\sigma \in \mathbf{co}(\tau)$. Hence the matrix unit ball here is operator compact.

Defining an analogue of total boundedness for operator spaces runs into immediate difficulties, since we only really know what balls centered at the origin look like. However we can avoid this by noting that a set K is totally bounded if K is bounded and if for every $\varepsilon > 0$ there is a finite dimensional subspace V_ε such that every point of K lies within ε of a point in V_ε.

This implies total boundedness: given any $\varepsilon > 0$, we can find a finite dimensional subspace $V_{\varepsilon/3}$ so that for every $x \in K$ there is a point $v \in V_{\varepsilon/3}$ such that $\|x - v\| < \varepsilon/3$. Since $V_{\varepsilon/3}$ is finite dimensional and K is closed and bounded, we can cover

$$S = \{v \in V_{\varepsilon/3} : d(K, v) < \varepsilon/3\}^-$$

by finitely many $\varepsilon/3$ balls, centered at $v_1, \ldots, v_k \in S$. But then for any $x \in K$, we can find a $v \in S$ so that $\|x - v\| \leq \varepsilon/3$ and there is an i so that v_i lies within $\varepsilon/3$ of v, and hence x lies within ε of one of the v_i, and so K is totally bounded.

Conversely, if K is totally bounded, for any $\varepsilon > 0$, choose $v_1, \ldots, v_n$ be the centers of ε-balls which cover K. Then

$$V_\varepsilon = \mathrm{span}\{v_1, \ldots, v_n\}$$

is a finite dimensional subspace which meets our criterion.

Hans Saar, a student of Wittstock, in his thesis [35] implicitly noticed this. He worked with compact maps, but if you look at his conditions on the maps, they imply the following about the images of unit balls.

Definition 4.1.2
A matrix point *$v = (v_n)$ in a vector space V is a sequence of points $v_i \in M_n(V)$ for $i \in \mathbb{N}$.*

A matrix set $\boldsymbol{K}$ in an operator space V is said to be strongly operator compact *if $\boldsymbol{K}$ is closed, completely bounded and if for all $\varepsilon > 0$, there exists a finite dimensional subspace V_ε of V such that for every matrix point $(x_n) \in \boldsymbol{K}$ we have a matrix point $(v_n) \in \boldsymbol{V}_\varepsilon$, such that $\|x_n - v_n\|_n < \varepsilon$ for all n.*

I would like to thank Zhong-Jin Ruan for bringing Saar's work to my attention and for providing this definition. We note that, as we will soon see, strong operator compactness is a *weaker* condition than operator compactness. The terminology was chosen to highlight the association between this definition and the strong operator approximation property, which *is* stronger than the operator approximation property. For those who find this terminology philosophically unpalatable, we suggest that *completely compact* may be an acceptable substitute, and is in harmony with Saar's work.

In Section 3.4 we put forward the idea that a matrix set should be compact if it is compact at each level. We used this substantially in Section 3.8 to develop the Arens-Mackey theory for local operator spaces. This is also the definition that was used in [41] to prove a version of the Krein-Milman theorem. We will call this condition *matrix compactness.*

The immediate and obvious question is: given these definitions do they agree, or under what conditions on V do they agree?

A matrix set which is operator compact is automatically strongly operator compact. First we note that $\overline{\mathbf{co}(x)}$ is strongly operator compact, since for any $\varepsilon > 0$, we choose n sufficiently large that $\|x - p_n x p_n\| \leq \varepsilon$ (we can do this since $\mathcal{K}(V)$ is the completion of $M_\infty(V)$) and let V_ε be the subspace spanned by the entries of $p_n x p_n$. Then given any matrix point $(v_1, v_2, \dots)$ in $\mathbf{co}(x)$ we let $v_i = (\sigma_i \otimes \mathrm{id})(x)$ and so if we let $v_i' = (\sigma_i \otimes \mathrm{id})(p_n x p_n)$, we have

$$\|v_i' - v_i\| = \|(\sigma_i \otimes \mathrm{id})(p_n x p_n - x)\| \leq \|(p_n x p_n - x)\| \leq \varepsilon.$$

Taking closures we get that any point in $\overline{\mathbf{co}(x)}$ must lie within ε of V_ε. We extend this to arbitrary operator compact sets $\boldsymbol{K} \subseteq \overline{\mathbf{co}\, x}$ by using the V_ε that you use for $\mathbf{co}\, x$. The converse is false in general, but the proof is highly non-trivial and we must wait until Lemma 4.4.3 in Section 4.4 before we can prove it. We will investigate conditions under which the two definitions agree in Section 4.5.

Strong operator compactness implies matrix compactness, since for each level K_n of a strongly operator compact set $\boldsymbol{K}$, and for every $\varepsilon > 0$, we have that every element of K_n lies within ε of the finite dimensional subspace $M_n(V_\varepsilon)$, and so K_n is closed and totally bounded.

The converse is not true in general. If we take $V = \ell^2(\mathbb{N})_c$, with standard basis $\{e_k\}$ and let X_n be the unit ball of $M_n(\mathrm{span}\{e_1, \dots, e_n\})$. Then this set is matrix compact, but is not strongly matrix compact, since given any finite dimensional subspace of V, we can find an e_m such that $d(e_m, V) > 1 - \varepsilon$. A natural question, to which the answer is unknown, would be whether the converse is true when we consider only matrix compact convex sets.

To summarize:

$$\text{operator compact} \implies \text{strongly operator compact} \implies \text{matrix compact}. \tag{4.1}$$

One direction that we wish to head with compactness is to look at the analogues of compact maps. As we mentioned, this is the area of theory which Saar was looking at, and one question that we will hope to explore is the relationship between the operator approximation property and approximations of compact maps.

Definition 4.1.3
If V, W are operator spaces and $\varphi \in \mathcal{CB}(V, W)$, then we say that if φ maps the matrix unit ball of V into

i. an operator compact set, then φ is operator compact,

ii. a strongly operator compact set, then φ is strongly operator compact,

iii. a matrix compact set, then φ is matrix compact.

We immediately notice that if φ is matrix compact then φ is clearly a compact map.

By (4.1) and the above comment, we have that we have the following implications for a map in $\mathcal{CB}(V,W)$:

operator compact $\Longrightarrow$ strongly operator compact $\Longrightarrow$ matrix compact $\Longrightarrow$ compact

We notice that operator compactness is stable under composition from both the left and the right, since if $\varphi \in \mathcal{CB}(V,W)$, $\psi \in \mathcal{CB}(W,X)$, then if $\varphi(\boldsymbol{B}^V) \subseteq \overline{\mathbf{co}\, w}$, we have

$$\psi(\varphi(\boldsymbol{B}^V)) \subseteq \overline{\mathbf{co}\, \psi_\infty(w)},$$

and if $\psi(\boldsymbol{B}^W) \subseteq \overline{\mathbf{co}\, x}$ then

$$\psi(\varphi(\boldsymbol{B}^V)) \subseteq \overline{\mathbf{co}\, \|\varphi\| x}.$$

This means that in particular the operator compact maps in $CB(V,V)$ form a two-sided ideal. It is an open question as to whether the space of operator compact maps must be closed in the cb-norm topology.

Similarly, matrix compact maps satisfy the above conditions in $\mathcal{CB}(V,W)$.

Saar showed in [35] that for strongly operator compact maps, all the above statements hold, and that they form a cb-norm closed subspace of $\mathcal{CB}(V,W)$.

4.2 C*-Operator Spaces

Just as looking at M_∞-convexity and M_∞-bimodules was a useful tool in the last two chapters, B. Johnson's C*-operator spaces will help us get a handle of some of the aspects of operator compactness. In particular, they will be crucial in our analysis of the operator approximation property.

If A is a C*-algebra then Johnson, in [20], defines an *A-operator space* $\mathcal{V}$ to be an essential A-bimodule with a norm which is absolutely A-convex, i.e. if v_1, v_2 lie in the unit ball of $\mathcal{V}$ then so does

$$v = \alpha_1 v_1 \beta_1 + \alpha_2 v_2 \beta_2 \tag{4.2}$$

where $\|\alpha_1\alpha_1^* + \alpha_2\alpha_2^*\| \leq 1$ and $\|\beta_1^*\beta_1 + \beta_1^*\beta_1\| \leq 1$.

The natural morphisms are the continuous bi-A-linear maps, i.e. continuous maps $\varphi : \mathcal{V} \to \mathcal{W}$ such that $\varphi(\alpha v \beta) = \alpha\varphi(v)\beta$ for all $v \in V$, α, $\beta \in A$.

For the purposes of this paper, we will only be using $\mathcal{K}$-operator spaces. In this special case we can rephrase our convexity axiom in much more familiar language. A set X in a $\mathcal{K}$-operator space is $\mathcal{K}$-convex if and only if it satisfies

(A$\mathcal{K}$C1) $\quad v + w \in X \quad$ for all orthogonal v, $w \in X$,

(A$\mathcal{K}$C2) $\quad \alpha v \beta \in X \quad$ for all $v \in X$, α, $\beta \in \mathcal{K}$, $\|\alpha\|$, $\|\beta\| \leq 1$.

We extend the definition of orthogonal in this context to: $v_1, \ldots, v_n \in \mathcal{V}$ are *orthogonal* if there exist orthogonal projections $e_1, \ldots, e_n \in \mathcal{K}$ such that $e_i v_i e_i = v_i$. A $\mathcal{K}$-norm is a norm $\|\cdot\|$ which satisfies

(A$\mathcal{K}$N1) $\quad \|v + w\| = \max\{\|v\|, \|w\|\} \quad$ for all orthogonal v, $w \in X$,

(A$\mathcal{K}$N2) $\quad \|\alpha v \beta\| \leq \|\alpha\|\|v\|\|\beta\| \quad$ for all $v \in X$, α, $\beta \in \mathcal{K}$.

Clearly the unit balls of $\mathcal{K}$-norms are $\mathcal{K}$-convex.

A norm which satisfies (4.2) is clearly $\mathcal{K}$-convex: if v and w are orthogonal and sit in the unit ball, then

$$v + w = e_1 v e_1 + e_2 v e_2$$

and

$$\|e_1 e_1^* + e_2 e_2^*\| = \|e_1^* e_1 + e_2^* e_2\| \leq 1$$

so $v + w$ lies in the unit ball; and if α, v, β are as in (A$\mathcal{K}$C2) then $\|\alpha\alpha^*\| = \|\alpha\|^2 \leq 1$ and similarly $\|\beta^*\beta\| \leq 1$, so $\alpha v \beta$ lies in the unit ball.

Conversely assume we have a $\mathcal{K}$-convex norm, and v_1, v_2 in its unit ball, and α_1, α_2, β_1, β_2 such that $\|\alpha_1\alpha_1^* + \alpha_2\alpha_2^*\| \leq 1$ and $\|\beta_1^*\beta_1 + \beta_1^*\beta_1\| \leq 1$. Then if we choose two isometries a and $b : \ell^2 \to \ell^2$ with orthogonal ranges H_1 and H_2, then we have

$$\begin{aligned} v &= \alpha_1 v_1 \beta_1 + \alpha_2 v_2 \beta_2 = \alpha_1 a^* a v_1 a^* a \beta_1 + \alpha_2 b^* b v_2 b^* b \beta_2 \\ &= \begin{bmatrix} \alpha_1 a^* & \alpha_2 b^* \end{bmatrix} \begin{bmatrix} a v_1 a^* & 0 \\ 0 & b v_2 b^* \end{bmatrix} \begin{bmatrix} a\beta_1 \\ b\beta_2 \end{bmatrix} \end{aligned}$$

But

$$\bar{v} = \begin{bmatrix} a v_1 a^* & 0 \\ 0 & b v_2 b^* \end{bmatrix} \in \mathcal{K}(H_1 \oplus H_2, H_1 \oplus H_2) \subseteq \mathcal{K}$$

and then we see that $\bar{v} = a v_1 a^* + b v_2 b^*$, and these are orthogonal, so $\bar{v}$ lies inside the unit ball. Similarly

$$\bar{\alpha} = \begin{bmatrix} \alpha_1 a^* & \alpha_2 b^* \end{bmatrix} \in \mathcal{K}(H_1 \oplus H_2, \ell^2) \subseteq \mathcal{K}$$

and

$$\bar{\beta} = \begin{bmatrix} a\beta_1 \\ b\beta_2 \end{bmatrix} \in \mathcal{K}(\ell^2, H_1 \oplus H_2) \subseteq \mathcal{K}$$

and we note that

$$\|\bar{\alpha}\| = \|\bar{\alpha}\bar{\alpha}^*\|^{1/2} = \|\alpha_1\alpha_1^* + \alpha_2\alpha_2^*\| \leq 1$$

and similarly for $\bar{\beta}$, so we have that $v = \bar{\alpha}\bar{v}\bar{\beta}$ lies in the unit ball.

If V is an operator space, we note that $\mathcal{K}(V) = \mathcal{K} \check{\otimes}_{\text{op}} V$, is a $\mathcal{K}$-operator space where $\mathcal{K}$ acts via

$$\alpha v \beta = [\sum_{j,k} \alpha_{i,j} v_{j,k} \beta_{k,l}]_{i,l},$$

or alternatively

$$\alpha(a \otimes v)\beta = \alpha a \beta \otimes v.$$

That the norm is in fact a $\mathcal{K}$-norm follows from the fact that $\mathcal{K}(V)$ is the completion of $M_\infty(V)$, and so (A$\mathcal{K}$N1) and (A$\mathcal{K}$N2) follow immediately from (AM_∞G1) and (AM_∞G2), respectively. Moreover, if $\varphi : V \to W$ is completely bounded, then

$$\varphi_\infty : \mathcal{K}(V) \to \mathcal{K}(W)$$

is bounded, since we know it is bounded from $M_\infty(V) \to M_\infty(W)$, and so it must extend to $\mathcal{K}(V)$.

Hence $\mathcal{K} : V \mapsto \mathcal{K}(V)$ takes operator spaces to $\mathcal{K}$-operator spaces, and $\mathcal{K} : \varphi \mapsto \varphi_\infty$ takes completely bounded maps to continuous bi-$\mathcal{K}$-linear maps. In other words $\mathcal{K}$ is a functor.

We define

$$\mathcal{F}(\mathcal{V}) = \{e_{1,1} v e_{1,1} : v \in \mathcal{V}\}.$$

Then $V = \mathcal{F}(\mathcal{V})$ is an operator space when we identify $M_n(V)$ with

$$\{p_n v p_n : v \in \mathcal{V}\}$$

via the map

$$\tau_n : [v_{i,j}] \to \sum_{i,j=1}^{n} e_{i,1} v_{i,j} e_{1,j},$$

and give it the norm inherited from $\mathcal{V}$. That this is in fact an operator space norm follows immediately from (A$\mathcal{K}$N1) and (A$\mathcal{K}$N2). If

$$\varphi : \mathcal{V} \to \mathcal{W}$$

is a continuous bi-$\mathcal{K}$-linear then $\varphi|_V$ is a completely bounded map from V to $W = \mathcal{F}(\mathcal{W})$, since

$$(\varphi|_V)_n([v_{i,j}]) = \varphi|_{p_n \mathcal{V} p_n}(\tau_n([v_{i,j}])).$$

Hence $\mathcal{F}$ is a functor from $\mathcal{K}$-operator spaces to operator spaces.

Proposition 4.2.1
The functors $\mathcal{K}$ and $\mathcal{F}$ implement an equivalence of categories between the category of operator spaces with completely bounded maps and the category of $\mathcal{K}$-operator spaces with continuous bi-$\mathcal{K}$-linear maps.

Proof :
We have an isomorphism of operator spaces $\iota_V : \mathcal{F}(\mathcal{K}(V)) \to V$ given by $[v_{i,j}] \to v_{1,1}$ (noting that all other entries are 0): this is clearly an isomorphism of vector spaces, and if $v = [v_{k,l}] \in M_n(\mathcal{F}(\mathcal{K}(V)))$, then

$$||\iota_{V,n}(v)||_n = ||[\iota_V([v_{k,l}])]||_n = ||[[v_{k,l}]_{1,1}]||_n$$

but

$$||v||_n = ||[[v_{k,l}]_{1,1}]|| = \sup_m ||[[v_{k,l}]_{1,1}]||_m = ||[[v_{k,l}]_{1,1}]||_n$$

and so ι_V is a completely isometric isomorphism. Furthermore it is a natural transformation, since if $\varphi \in \mathcal{CB}(V, W)$, then we have that

$$\iota_W(\mathcal{F}(\mathcal{K}(\varphi))([v_{i,j}])) = \varphi(v_{1,1}) = (\mathcal{F}(\mathcal{K}(\varphi))(\iota_V([v_{i,j}])$$

So we have a natural equivalence $\mathcal{F} \circ \mathcal{K} \cong \mathrm{id}$.

Similarly we have an isomorphism of $\mathcal{K}$-operator spaces $\tau : \mathcal{K}(\mathcal{F}(\mathcal{V})) \to \mathcal{V}$ given by $(v_{i,j}) \to \sum e_{i,1} v_{i,j} e_{1,j}$, which again is bijective, is bi-$\mathcal{K}$-linear, since if $v = (v_{i,j}) \in \mathcal{K}(\mathcal{F}(\mathcal{V}))$, then

$$\tau(\alpha v \beta) = \sum_{i,j,k,l} e_{i,1} \alpha_{i,k} v_{k,l} \beta_{l,j} e_{1,j} = \sum_{k,l} (\alpha e_{k,1}) v_{k,l} (e_{1,l} \beta) = \alpha \tau(v) \beta,$$

and is an isometric isomorphism since

$$\begin{aligned} ||\tau(v)|| &= ||\sum_{i,j} e_{i,1} v_{i,j} e_{1,j}|| = \sup_n ||p_n(\sum_{i,j} e_{i,1} v_{i,j} e_{1,j}) p_n|| = \sup_n ||\sum_{i,j=1}^{n} e_{i,1} v_{i,j} e_{1,j}|| \\ &= ||v||. \end{aligned}$$

Again, this is a natural transformation, and so we have a natural equivalence $\mathcal{K} \circ \mathcal{F} \cong \mathrm{id}$.

Hence the two categories are equivalent. □

Note that there are many possible ways of implementing this equivalence.

The point of proving such a formal result, is that we can quickly transfer results from the theory of operator spaces to $\mathcal{K}$-operator spaces and vice-versa. In particular, it allow us to quickly prove a Hahn-Banach theorem for $\mathcal{K}$-operator spaces.

Proposition 4.2.2
If $\mathcal{V}$, $\mathcal{W}$ are $\mathcal{K}$-operator spaces such that $\mathcal{V}$ is a bi-$\mathcal{K}$-invariant subspace of $\mathcal{W}$, then given any continuous bi-$\mathcal{K}$-linear functional

$$\varphi : \mathcal{V} \to \mathcal{K}$$

then there is a continuous bi-$\mathcal{K}$-linear functional

$$\bar{\varphi} : \mathcal{W} \to \mathcal{K}$$

such that $\bar{\varphi}|_{\mathcal{V}} = \varphi$ and $||\bar{\varphi}|| = ||\varphi||$.

Proof :

We have shown that we can find $\psi = \mathcal{F}(\varphi)$ which is a completely bounded linear functional on $V = \mathcal{F}(\mathcal{V})$. The bi-$\mathcal{K}$-invariance of $\mathcal{V}$ in $\mathcal{W}$ implies that V is a subspace of $W = \mathcal{F}(\mathcal{W})$, and so we get by the Hahn-Banach theorem for absolute gauges (Theorem 2.3.1) that there is a completely bounded linear functional $\bar{\psi}$ on W extending ψ. We then push this back to the original category to get a bi-$\mathcal{K}$-linear functional $\bar{\varphi}$ which extends φ. It remains only to note that the norms are preserved by the functors. □

This correspondence of categories is preserves the appropriate sense of convexity. If V is a vector space and $\boldsymbol{X} \subseteq \boldsymbol{V}$ is a matrix convex set, let ${}_{\boldsymbol{X}}V$ be V with the operator Minkowski seminorm of $\boldsymbol{X}$. Then the unit ball of $\mathcal{K}({}_{\boldsymbol{X}}V)$ with the corresponding $\mathcal{K}$-seminorm determines a $\mathcal{K}$-convex set X. Conversely, given a $\mathcal{K}$-convex set we can get a corresponding matrix convex set. More concretely, the correspondence is given by

$$X = \{v \in \mathcal{K}(V) : \pi_n(v) \in X_n,\ \forall n \in \mathbb{N}\} \tag{4.3}$$

and

$$\boldsymbol{X} = (\pi_n(X)) \tag{4.4}$$

where $\pi_n(x) = p_n x p_n \subseteq M_n(V)$

Our immediate concern is with operator compact sets. We would like to find a $\mathcal{K}$ version of the sequence version of compactness, in particular.

If $\mathcal{V}$ is a $\mathcal{K}$-operator space, then we define the *absolutely $\mathcal{K}$-convex hull*, $\mathrm{co}_{\mathcal{K}}\, X$, of a set $X \subseteq \mathcal{V}$ to be

$$\{v \in \mathcal{V} : v = \sum_{i=1}^{n} \alpha_i x_i \beta_i, x_i \in X, \alpha_i, \beta_i \in \mathcal{K}, \|\sum_{i=1}^{n} \alpha_i^* \alpha_i\|, \|\sum_{i=1}^{n} \beta_i^* \beta_i\| \leq 1\},$$

or, equivalently, the smallest $\mathcal{K}$-convex set containing X.

Definition 4.2.1
Let $\mathcal{V}$ be a $\mathcal{K}$-operator space. We say that a set $X \subseteq \mathcal{V}$ is $\mathcal{K}$-compact *if there is a sequence $\{x_i\} \subset \mathcal{V}$ which converges to 0, such that X is closed and $X \subseteq \overline{\mathrm{co}_{\mathcal{K}}\, \{x_i\}}$.*

If we restrict our attention to matrix convex and $\mathcal{K}$-convex sets, we can compare $\mathcal{K}$-compactness with the other definitions. In particular, we have that $\mathcal{K}$-compactness of a $\mathcal{K}$-convex set X implies operator compactness in the corresponding matrix convex set $\boldsymbol{X}$. To see this, we assume that

$$X \subseteq \overline{\mathrm{co}_{\mathcal{K}}(\{x_1, x_2, \dots\})}.$$

Then we let λ be an isomorphism of $\mathcal{K} \check{\otimes} \mathcal{K}$ with $\mathcal{K}$, where, for simplicity's sake, we will assume λ is induced by a bijection $\mu : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ via

$$\lambda(e_{(i,j),(k,l)}) = e_{\mu(i,j),\mu(k,l)}$$

and let

$$x = \lambda(\mathrm{diag}(x_1, x_2, \dots)).$$

Then if $v \in \pi_n(\mathrm{co}_{\mathcal{K}}(\{x_1, x_2, \dots\}) \subseteq M_n(V)$ is given by

$$v = p_n \sum_{i=1}^{k} \alpha_k x_k \beta_k p_n$$

we consider the map $\phi : \mathcal{K} \check{\otimes} \mathcal{K} \to M_n$ given by

$$w \mapsto \begin{bmatrix} p_n\alpha_1 & p_n\alpha_2 & \cdots & p_n\alpha_k & 0 & \cdots \end{bmatrix} w \begin{bmatrix} \beta_1 p_n \\ \beta_2 p_n \\ \vdots \\ \beta_k p_n \\ 0 \\ \vdots \end{bmatrix}$$

so that $v = \phi(\mathrm{diag}(x_1, x_2, \ldots)) = \phi(\lambda^{-1}(x))$. We note that But $phi \circ \lambda^{-1} : \mathcal{K} \to M_m$ is given by

$$w \mapsto [a_{i,j}]w[b_{i,j}]$$

where

$$a_{\mu(i,j),\mu(s,t)} = \begin{cases} \alpha_{i,j}, & \text{if } i = 1,\ j \leq n,\ s \leq k \\ 0, & \text{otherwise,} \end{cases}$$

and

$$b_{\mu(i,j),\mu(s,t)} = \begin{cases} \alpha_{i,j}, & \text{if } s = 1,\ t \leq n,\ i \leq k \\ 0, & \text{otherwise,} \end{cases}$$

and $m = \max \mu(1, \{1, \ldots, n\})$. So $\phi \circ \lambda^{-1}$ is an element of $\mathcal{CB}(\mathcal{K}, M_n)$ or, rewriting, an element of $M_n(\mathcal{T})$, and $\|\phi \circ \lambda^{-1}\|_{\mathcal{T}} \leq 1$. So the image of $\mathrm{co}_K(\{x_1, x_2, \ldots\})$ sits inside $\overline{\mathbf{co}(x)}$. But π_n is continuous, so if $v_\nu \in \mathrm{co}_K(\{x_1, x_2, \ldots\})$ converges to

$$v \in \overline{\mathrm{co}_K(\{x_1, x_2, \ldots\})}$$

then $\pi_n(v_\nu) \to \pi_n(v) \in \overline{\mathbf{co}(x)}$. Hence $\boldsymbol{X} \subseteq \overline{\mathbf{co}(x)}$.

So we have that across the identification of categories

$$\mathcal{K}\text{-compactness} \implies \text{operator compactness}$$

Turning to look at mappings, if $\mathcal{V}$, $\mathcal{W}$ are $\mathcal{K}$-operator spaces, and $\varphi \in \mathcal{KB}(\mathcal{V}, \mathcal{W})$ maps the unit ball of $\mathcal{V}$ into a $\mathcal{K}$-compact set in $\mathcal{W}$, then we say that φ is *$\mathcal{K}$-compact*.

Given that $\mathcal{K}$-compact convex sets give rise to matrix convex sets, and the fact that the images of unit balls will be convex in the appropriate sense, then it is clear that under the equivalence of categories $\mathcal{K}$-compact mappings become operator compact mappings. Furthermore, since operator compact sets are strongly operator compact, any strongly operator compact operator is operator compact.

4.3 Approximation Properties

In their article on operator approximation properties [9], Effros and Ruan observed that classically the following notions of convergence all agree. Let V and W be Banach spaces, and let φ_ν a bounded net of maps in $\mathcal{B}(V, W)$, and φ in $\mathcal{B}(V, W)$. The following are equivalent:

i. φ_ν converges to φ uniformly on compact sets in V,

ii. $\varphi_\nu \otimes \mathrm{id} : V \check{\otimes} c_0 \to W \check{\otimes} c_0$ converges point-norm to $\varphi \otimes \mathrm{id}$,

iii. $\varphi_\nu \otimes \mathrm{id} : V \check{\otimes} X \to W \check{\otimes} X$ converges point-norm to $\varphi \otimes \mathrm{id}$, for any Banach space X,

iv. $\varphi_\nu \otimes \mathrm{id} : V \check{\otimes} \ell^\infty(S) \to W \check{\otimes} \ell^\infty(S)$ converges point-norm to $\varphi \otimes \mathrm{id}$ for any set S.

They then noted that in the category of operator spaces there were the following analogues to the above. Let V and W be operator spaces, and let φ_ν a bounded net of maps in $\mathcal{CB}(V, W)$, and φ in $\mathcal{CB}(V, W)$.

ii. $\varphi_\nu \otimes \mathrm{id} : V \check{\otimes}_{\mathrm{op}} c_0 \to W \check{\otimes}_{\mathrm{op}} c_0$ converges point-norm to $\varphi \otimes \mathrm{id}$,

iii. $\varphi_\nu \otimes \mathrm{id} : V \check{\otimes}_{\mathrm{op}} X \to W \check{\otimes}_{\mathrm{op}} X$ converges point-norm to $\varphi \otimes \mathrm{id}$, for any operator space X,

iv. $\varphi_\nu \otimes \mathrm{id} : V \check{\otimes}_{\mathrm{op}} \mathcal{B}(H) \to W \check{\otimes}_{\mathrm{op}} \mathcal{B}(H)$ converges point-norm to $\varphi \otimes \mathrm{id}$ for any Hilbert space H.

Unlike the Banach space case, these are not all equivalent. That (iii) and (iv) are equivalent and imply (ii) was recognized by Effros and Ruan. That (ii) does not imply (iii) or (iv) is due to some deep results of Kirchberg (see Section 4.4 for a more detailed discussion). There was no obvious operator space analogue of (i) and they wrote [9]:

> It would be of considerable interest to find an analogue of (i) for operator spaces. A related problem is to formulate an operator space version of Grothendieck's result that a Banach space V has the approximation property if and only if any compact operator $K : W \to V$ is a uniform limit of finite rank operators.

The objective of this section is to solve these two problems.

We begin by postulating the operator space analogue of (i), since we now have a number of candidates for compact sets. The correct choice for our convergence result is operator compactness. We will discuss what happens if you replace operator compact sets by strongly operator compact sets in Section 4.4.

Definition 4.3.1
Let V, W be operator spaces, φ_ν, $\varphi \in \mathcal{CB}(V, W)$. We say that the net φ_ν converges to φ completely uniformly on operator compact sets *if for all operator compact sets $\boldsymbol{X} \subset V$ and $\varepsilon > 0$ there is an N such that*

$$\sup\{\|(\varphi_\nu)_n(x) - \varphi_n(x)\|_n : x \in X_n,\ n \in \mathbb{N}\} < \varepsilon,$$

for all $\nu \geq N$.

Our aim is to show that this topology is the same as the topology (ii), which we call the *stable point-norm topology*. This above choice does in fact give us the result we expect. In fact we get associated results for convergence on $\mathcal{K}$-compact sets in $\mathcal{K}$-operator algebras.

Proposition 4.3.1
Let V, W be operator spaces, φ_ν, $\varphi \in \mathcal{CB}(V, W)$. Then the following are equivalent:

i. $\varphi_\nu \to \varphi$ in the stable point norm topology on $\mathcal{CB}(V, W)$.

ii. $\varphi_\nu \to \varphi$ completely uniformly on operator compact sets of V.

iii. $\mathrm{id} \otimes (\varphi_\nu)_\infty \to \mathrm{id} \otimes \varphi_\infty$ point-norm on $\mathcal{B}_{\mathcal{K}}(c_0(\mathcal{K}(V)), c_0(\mathcal{K}(V)))$.

iv. $(\varphi_\nu)_\infty \to \varphi_\infty$ uniformly on the compact sets of $\mathcal{K}(V)$.

v. $(\varphi_\nu)_\infty \to \varphi_\infty$ uniformly on the $\mathcal{K}$-compact sets of $\mathcal{K}(V)$.

However to prove this, we will need the following lemma about tensor products of commutative C*-algebras with operator spaces.

Lemma 4.3.2
If V is an operator space, and X is a locally compact Hausdorff space, then $C_0(X) \check{\otimes} V \cong C_0(X) \check{\otimes}_{op} V$ as Banach spaces.

Proof (Lemma 4.3.2):

Recall that there exists a C*-algebra A such that V embeds completely isometrically in A, and that $C_0(X) \check{\otimes} A \cong C_0(X) \check{\otimes}_{\mathrm{op}} A$ as Banach spaces. However, the minimal tensor products respect inclusion, and so $C_0(X) \underset{\smile}{\otimes} V$ and $C_0(X) \underset{\smile}{\otimes}_{\mathrm{op}} V$ are isometrically isomorphic as normed vector spaces, and so their completions agree. □

Proof (Proposition 4.3.1):

(i $\implies$ ii): Given $\boldsymbol{X}$ operator compact and $\varepsilon > 0$, let $x \in \mathcal{K}(V)$ such that $\boldsymbol{X} \subseteq \overline{\mathbf{co}\, x}$. Then for any ν such that

$$\|(\varphi_\nu \otimes \mathrm{id})(x) - (\varphi \otimes \mathrm{id})(x)\| < \frac{\varepsilon}{\|x\|}$$

we have

$$\begin{aligned}
&\sup\{\|\boldsymbol{\varphi_\nu}(v) - \boldsymbol{\varphi}(v)\|_n : \forall n, v \in \mathbf{co}\, x_n\} \\
&= \sup\{\|(\varphi_\nu)_n((\sigma \otimes \mathrm{id})(x)) - \varphi_n((\sigma \otimes \mathrm{id})(x))\|_\infty : \sigma \in M_n(\mathcal{F}),\ \|\sigma\|_{\mathcal{T},n} \leq 1\} \\
&= \sup\{\|(\sigma \otimes \mathrm{id})((\varphi_\nu)_n(x)) - \varphi_n(x))\|_\infty : \sigma \in M_n(\mathcal{F}),\ \|\sigma\|_{\mathcal{T},n} \leq 1\} \\
&\leq \|\boldsymbol{\varphi_\nu}(x) - \boldsymbol{\varphi}(x)\|_\infty \\
&= \|(\varphi_\nu \otimes \mathrm{id})(x) - (\varphi \otimes \mathrm{id})(x)\| \\
&< \varepsilon
\end{aligned}$$

and so by a simple continuity argument, for ν sufficiently large we have

$$\sup_{v \in \boldsymbol{X}} \|\boldsymbol{\varphi_\nu}(v) - \boldsymbol{\varphi}(v)\|_\infty < \varepsilon$$

(ii $\implies$ i): Given $x \in \mathcal{K}(V)$ we notice that $\pi_n(x) \in \mathbf{co}\, x$ for all n, that $\mathbf{co}\, x$ is operator compact, and that

$$\begin{aligned}
\|(\varphi_\nu \otimes \mathrm{id})(x) - (\varphi \otimes \mathrm{id})(x)\| &= \sup_n \|\pi_n(\boldsymbol{\varphi_\nu}(x) - \boldsymbol{\varphi}(x))\|_n \\
&= \sup_n \|\boldsymbol{\varphi_\nu}(\pi_n(x)) - \boldsymbol{\varphi}(\pi_n(x))\|_\infty \\
&< \varepsilon
\end{aligned}$$

for ν sufficiently large.

(i $\implies$ iii): We observe that

$$\mathcal{K}(V) \cong \mathcal{K} \check{\otimes}_{\mathrm{op}} V \cong \mathcal{K} \check{\otimes}_{\mathrm{op}} \mathcal{K} \check{\otimes}_{\mathrm{op}} V \supseteq c_0 \check{\otimes}_{\mathrm{op}} \mathcal{K} \check{\otimes}_{\mathrm{op}} V \cong c_0 \check{\otimes} (\mathcal{K} \check{\otimes}_{\mathrm{op}} V)$$

as Banach spaces by Lemma 4.3.2. So if $\varphi_\nu \to \varphi$ in the stable point norm topology, then

$$\mathrm{id} \otimes \mathrm{id} \otimes \varphi_\nu \to \mathrm{id} \otimes \mathrm{id} \otimes \varphi$$

in the point-norm topology on $c_0 \check{\otimes} \mathcal{K}(V)$.

(iii $\iff$ iv): This is the classical result, since $\mathcal{K}(V)$, $\mathcal{K}(W)$ are Banach spaces.

(iv $\implies$ i): Follows since points are compact.

(iii $\implies$ v): If X is any $\mathcal{K}$-compact set, $X \subseteq \overline{\mathrm{co}_{\mathcal{K}}\,\{x_i\}}$, then

$$\begin{aligned}
&\sup_{v \in \mathrm{co}_{\mathcal{K}}\,\{x_i\}} \|(\varphi_\nu)_\infty(v) - \varphi_\infty(v)\| \\
&\leq \sup\{\|(\varphi_\nu)_\infty(\sum_{i=1}^n \alpha_i x_i \beta_i) - \varphi_\infty(\sum_{i=1}^n \alpha_i x_i \beta_i)\| : \\
&\qquad \|\sum_{i=1}^n \alpha_i^* \alpha_i\|,\ \|\sum_{i=1}^n \beta_i \beta_i^*\| \leq 1, n \in \mathbb{N}\} \\
&\leq \sup\{\|\sum_{i=1}^n \alpha_i((\varphi_\nu)_\infty(x_i) - \varphi_\infty(x_i))\beta_i\| : \\
&\qquad \|\sum_{i=1}^n \alpha_i^* \alpha_i\|,\ \|\sum_{i=1}^n \beta_i \beta_i^*\| \leq 1, n \in \mathbb{N}\} \\
&\leq \sup\{\|(\varphi_\nu)_\infty(x_i) - \varphi_\infty(x_i)\|\}
\end{aligned}$$

since balls are $\mathcal{K}$-convex. Again a simple continuity argument then gives the result.

(v $\implies$ iii): Follows since sequences converging to 0 are $\mathcal{K}$-compact. □

So we have answered the first part of the question.

Recall that a Banach space X has the *approximation property* if the identity map id : $X \to X$ can be approximated uniformly on compact sets by finite rank maps.

Theorem 4.3.3
Let X be a Banach space. The following are equivalent:

i. *X has the approximation property,*

ii. *For all Banach spaces Y, the finite rank maps are dense in $\mathcal{B}(Y, X)$ with the topology of uniform convergence on compact sets,*

iii. *For all Banach spaces Y, the finite rank maps are dense in $\mathcal{B}(X, Y)$ with the topology of uniform convergence on compact sets,*

iv. *If $\omega \in X^* \widehat{\otimes} X$ and $\omega(x) = 0$ for all $x \in X$ then $\tau(\omega) = 0$, where $\tau(\sum \chi_i \otimes x_i) = \sum \chi_i(x_i)$,*

v. *Given any Banach space Y, any compact map in $\mathcal{B}(Y, X)$ can be approximated uniformly by finite rank maps.*

Effros and Ruan reformulated the definition by noting that the equivalence of topologies described at the start of this section means that one can replace the topology of uniform convergence on compact sets by the stable point-norm topology.

Definition 4.3.2
V has the operator approximation property *if the identity map* id : $V \to V$ *can be approximated by completely bounded finite rank maps in the stable point norm topology.*

Given this definition, one can prove the following theorem, in direct analogy with Theorem 4.3.3.

Theorem 4.3.4
Let V be an operator space. The following are equivalent:

i. *V has the operator approximation property.*

ii. *For all operator spaces W, the finite rank maps in $\mathcal{CB}(W,V)$ are dense in the stable point norm topology.*

iii. *For all operator spaces W, the finite rank maps in $\mathcal{CB}(V,W)$ are dense in the stable point norm topology.*

iv. *If $\omega \in V^* \widehat{\otimes}_{op} V \subseteq \mathcal{CB}(V,V)$ and $\omega(v) = 0$ for all $v \in V$ then $\tau(\omega) = 0$, where τ is the* matricial trace $\tau(\omega) = \omega(\mathrm{id})$, $\mathrm{id} : V \to V$.

Our ultimate aim is to formulate an operator space version of (v). First we make a necessary detour. By a finite $\mathcal{K}$-rank map in $\mathcal{B}_{\mathcal{K}}(\mathcal{V},\mathcal{W})$ we mean a map whose range lies inside a finitely generated bi-$\mathcal{K}$-submodule of $\mathcal{W}$. The bi-$\mathcal{K}$-submodules of $\mathcal{V}$ are in one to one correspondence with the finite dimensional subspaces of $\mathcal{F}(\mathcal{V})$ under the categorical equivalence, so the finite $\mathcal{K}$-rank maps are in correspondence with the finite rank maps.

Definition 4.3.3
We say that a $\mathcal{K}$-operator space $\mathcal{V}$ has the $\mathcal{K}$-approximation property if the identity map can be approximated uniformly on $\mathcal{K}$-compact sets by finite $\mathcal{K}$-rank maps in $\mathcal{B}_{\mathcal{K}}(\mathcal{V},\mathcal{V})$.

This definition and Proposition 4.3.1 give us immediately the following corollary.

Corollary 4.3.5
An operator space V has the operator approximation property if and only if $\mathcal{K}(V)$ has the $\mathcal{K}$-approximation property.

We note that there is the expected $\mathcal{K}$ version of Theorem 4.3.4.

Theorem 4.3.6
The following conditions are equivalent to a $\mathcal{K}$-operator space $\mathcal{V}$ having the operator approximation property:

i. *For all $\mathcal{K}$-operator spaces $\mathcal{W}$, the finite $\mathcal{K}$-rank maps are dense in $\mathcal{B}_{\mathcal{K}}(\mathcal{W},\mathcal{V})$ with topology of uniform convergence on $\mathcal{K}$-compact sets.*

ii. *For all $\mathcal{K}$-operator spaces $\mathcal{W}$, the finite $\mathcal{K}$-rank maps are dense in $\mathcal{B}_{\mathcal{K}}(\mathcal{V},\mathcal{W})$ with topology of uniform convergence on $\mathcal{K}$-compact sets.*

Proof :
This result follows from the fact that finite $\mathcal{K}$-rank maps in $\mathcal{B}_{\mathcal{K}}(\mathcal{W},\mathcal{V})$ and $\mathcal{B}_{\mathcal{K}}(\mathcal{V},\mathcal{W})$ correspond to finite rank maps in $\mathcal{CB}(\mathcal{W},\mathcal{V})$ and $\mathcal{CB}(\mathcal{V},\mathcal{W})$ respectively, coupled with the equivalence of the two categories, and the correspondence between notions of convergence. □

We are now in the position to answer the second part of Effros and Ruan's question. In doing so the real reason for bringing in the machinery of $\mathcal{K}$-operator spaces becomes clear: by using $\mathcal{K}$-operator space methods, one can use the classical method of proof in the difficult direction.

Theorem 4.3.7
An operator space V has the operator approximation property, if and only if for any operator space W and any operator compact map $\varphi \in \mathcal{CB}(W,V)$, φ can be approximated completely uniformly by finite rank maps.

Proof :
There exists an $x \in \mathcal{K}(V)$ such that $\boldsymbol{\varphi}(\boldsymbol{B}^W) \subseteq \mathbf{co}\, x$. Since V has the approximation property, for any $\varepsilon > 0$ there is a finite rank map $\psi \in \mathcal{CB}(V,V)$ such that

$$\|(\psi \otimes \mathrm{id})(x) - x\| < \varepsilon.$$

Therefore

$$\|\psi(v) - v\|_\infty < \varepsilon$$

for all $v \in \mathbf{co}\, x$ and so

$$\|\psi \circ \varphi - \varphi\|_{cb} < \varepsilon.$$

To prove the converse we switch to the $\mathcal{K}$-operator space language. Since $\mathcal{K}$-compactness of φ_∞ implies operator compactness of φ, and if we can approximate φ completely uniformly by finite rank maps, then we can approximate φ_∞ by finite $\mathcal{K}$-rank maps.

Let X be any $\mathcal{K}$-compact set in $\mathcal{K}(V)$, and without loss of generality, we may assume that

$$X = \overline{\mathrm{co}_{\mathcal{K}}\, \{x_i\}}$$

for some sequence $\{x_i\} \in c_0(\mathcal{K}(V))$, and that $x_i \neq 0$ for all i. Now let

$$U = \overline{\mathrm{co}_{\mathcal{K}}\, \{\frac{x_i}{\|x_i\|^{1/2}}\}},$$

and so the identity map

$$\mathrm{id} : {}_U\mathcal{K}(V) \to V$$

is $\mathcal{K}$-compact. Therefore we can find finite $\mathcal{K}$-rank maps

$$\varphi_k : {}_U\mathcal{K}(V) \to \mathcal{K}(V)$$

which approximates id in norm; in particular, we may let each map be of the form

$$\varphi_k(v) = \sum_{i=1}^{n_k} \psi_{k,i}(v) v_{k,i}$$

where $\psi_{k,i} \in ({}_U\mathcal{K}(V))^*$. So all we need do is show we can approximate elements of $({}_U\mathcal{K}(V))^*$ by elements of $\mathcal{K}(V)^*$ uniformly on X, i.e. given $\delta > 0$, then if for all k, i we can find $\psi'_{k,i} \in \mathcal{K}(V)^*$ such that

$$\|\psi'_{k,i}(v) - \psi_{k,i}(v)\| < \delta$$

for all $v \in X$, then if we let

$$\varphi'_k(v) = \sum_{i=1}^{n_k} \psi'_{k,i}(v) v_{k,i}$$

we have

$$\|\varphi'_k(v) - v\| \leq \|\varphi'_k(v) - \varphi_k(v)\| + \|\varphi_k(v) - v\| < 2\delta$$

for all k sufficiently large, whence the result.

So given any $\psi \in ({}_U\mathcal{K}(V))^*$ we may assume $\|\psi\|_U = 1$ without loss of generality. We note that $\frac{x_i}{\|x_i\|^{1/2}} \in U$ for all i, and so we have that

$$\left\| \frac{x_i}{\|x_i\|^{1/2}} \right\|_U \leq 1$$

for all i. But this is equivalent to saying that

$$\|x_i\|_U \leq \|x_i\|^{1/2}$$

for all i, and so $\|x_i\|_U \to 0$ as $i \to \infty$. Now choose an N so that $\|x_i\| < \delta^2/2$ for all $i > N$. Then let

$$\mathcal{V}_\delta = \mathrm{span}_{\mathcal{K}}\, \{x_i\}_{i=1}^N,$$

and let ψ_δ be the restriction of ψ to $\mathcal{V}_\delta$. We note that since $\mathcal{V}_\delta$ is of finite $\mathcal{K}$-dimension and so we have that $\psi_\delta \in \mathcal{V}_\delta^*$ and

$$\|\psi_\delta\| \leq \|\psi_\delta\|_U = 1,$$

since U sits inside the unit ball of $\mathcal{K}V$. But by the Hahn-Banach theorem for $\mathcal{K}$-operator spaces, we can find a $\psi' \in \mathcal{V}^*$ which agrees with ψ_δ on $\mathcal{V}_\delta$ and

$$\|\psi'\| = \|\psi_\delta\| \leq 1.$$

Now for any x_i we have either that $\psi'(x_i) = \psi(x_i)$ (if $i \leq N$), or

$$\begin{aligned}\|\psi(x_i) - \psi'(x_i)\| &\leq \|\psi(x_i)\| + \|\psi'(x_i)\| \\ &\leq \|\psi\|\|x_i\|_U + \|\psi'\|\|x_i\| \\ &\leq \delta/2 + \delta^2/2 \leq \delta \text{ (for } \delta \text{ small).}\end{aligned}$$

and it is easy to see that this implies $\|\psi(x) - \psi'(x)\| < \delta$ for all $x \in X$. □

We note here that we could, without loss of generality, take the closure of the operator compact operators in the completely uniform topology on $\mathcal{CB}(W, V)$.

4.4 Strong Operator Approximation Properties

There remains the question of where the strong matrix compactness of Saar fits into the picture. Effros and Ruan in their original paper defined an operator space V as having the *strong operator approximation property* if the identity map id : $V \to V$ can be approximated by finite rank mappings φ_ν in the *strongly stable point-norm topology*, i.e. if $\varphi_\nu \otimes \mathrm{id} \to \mathrm{id} \otimes \mathrm{id}$ point-norm in $V \check{\otimes}_{\mathrm{op}} \mathcal{B}(\ell^2)$ (more generally for φ_ν, $\varphi \in \mathcal{CB}(V, W)$ for some V, W operator spaces, we say that $\varphi_\nu \to \varphi$ in the strongly stable point-norm topology if $\varphi_\nu \otimes \mathrm{id} \to \varphi \otimes \mathrm{id}$ in the point-norm topology on $\mathcal{CB}(V \check{\otimes}_{\mathrm{op}} \mathcal{B}(\ell^2), W \check{\otimes}_{\mathrm{op}} \mathcal{B}(\ell^2))$). Since $\mathcal{K} \check{\otimes}_{\mathrm{op}} V \hookrightarrow \mathcal{B}(H) \check{\otimes}_{\mathrm{op}} V$, it is easy to see that the strong operator approximation property implies the operator approximation property. Effros and Ruan posed the question as to whether or not the strong operator approximation property was equivalent to the operator approximation property, conjecturing that it was not.

Kirchberg [22, 23] showed that this is in fact the case. A sketch of the argument is as follows: the strong operator approximation property is the same as the general slice map property which, for C*-algebras, implies exactness. Extensions of C*-algebras with the operator approximation property have the operator approximation property, but there is an extension of cone $C_r^*(SL(2, \mathbb{Z}))$ by $\mathcal{K}$ (both of which have the operator approximation property) which is not exact, and so cannot have the strong operator approximation property. He also showed that if the operator space is locally reflexive, then they do agree.

We would like to gain some mileage in our situation from these facts. To do this we need to introduce the topology of completely uniform convergence of strongly compact sets:

Definition 4.4.1
If V and W are operator spaces, we say that a sequence of maps $\varphi_\nu \in \mathcal{CB}(V, W)$ converges to $\varphi \in \mathcal{CB}(V, W)$ completely uniformly on strongly operator compact sets *if for all strongly operator compact sets $\boldsymbol{X} \subset V$ and $\varepsilon > 0$ there is an N such that*

$$\sup\{\|(\varphi_\nu)_n(x) - \varphi_n(x)\|_n : x \in X_n,\ n \in \mathbb{N}\} < \varepsilon,\ \forall \nu \geq N$$

Clearly if $\varphi_\nu \to \varphi$ completely uniformly on strongly operator compact sets, then it converges completely uniformly on operator compact sets, and hence in the stable point-norm topology. I am are indebted to Zhong-Jin Ruan for pointing out the following:

Lemma 4.4.1
If φ_ν, $\varphi \in \mathcal{CB}(V, W)$ for some V, W operator spaces, and $\varphi_\nu \to \varphi$ completely uniformly on strongly operator compact sets then $\varphi_\nu \to \varphi$ in the strongly stable point-norm topology.

Proof :
Fix some $a \in V \check{\otimes}_{\mathrm{op}} \mathcal{B}(H)$, and some $\varepsilon > 0$. So for all $\eta > 0$ there is some $a_\eta = \sum_{i=1}^n \alpha_i \otimes v_i$ such that $||a - a_\eta|| < \eta$. Let $\{e_n\}$ be a basis for ℓ^2 and let p_n be the projection onto the span for the first n vectors. We let $a_n = p_n a p_n$ and note that $||a|| = \sup\{||a_n||\}$. So we consider the matrix set (a_n), and note that the matrix point $(p_n a_\eta p_n)$ is within ε of (a_n). But $(p_n a_\eta p_n) \in \mathrm{span}\{v_1, \ldots, v_n\} =: V_\eta$ which is a finite dimensional space. So (a_n) is strongly operator compact, so for ν sufficiently large, we have that $||(\varphi_\nu)_n(a_n) - \varphi_n(a_n)|| < \varepsilon/3$. Hence for ν and n sufficiently large

$$\begin{aligned} ||(\varphi_\nu \otimes \mathrm{id})(a) - (\varphi \otimes \mathrm{id})(a)|| &\leq ||(\varphi_\nu \otimes \mathrm{id})(a) - (\varphi_\nu \otimes \mathrm{id})(a_n)|| \\ &\quad + ||(\varphi_\nu)_n(a_n) - \varphi_n(a_n)|| \\ &\quad + ||(\varphi \otimes \mathrm{id})(a_n) - (\varphi \otimes \mathrm{id})(a)|| \\ &\leq \varepsilon \end{aligned}$$

□

We would like to prove a converse result. What we will prove is actually slightly stronger. We will say that $\varphi_\nu \in \mathcal{CB}(V, W)$ converges to $\varphi \in \mathcal{CB}(V, W)$ *completely uniformly on matrix compact sets* if for all matrix compact sets $\boldsymbol{X} \subset V$ and $\varepsilon > 0$ there is an N such that

$$\sup\{||(\varphi_\nu)_n(x) - \varphi_n(x)||_n : x \in X_n,\ n \in \mathbb{N}\} < \varepsilon,\ \forall \nu \geq N$$

Clearly by (4.1) we have that uniform convergence on matrix compact sets implies uniform convergence on strongly operator compact sets.

Lemma 4.4.2
If φ_ν, $\varphi \in \mathcal{CB}(V, W)$ for some V, W operator spaces, and $\varphi_\nu \to \varphi$ in the strongly stable point-norm topology then $\varphi_\nu \to \varphi$ completely uniformly on matrix compact sets.

Proof :
Let $\boldsymbol{X}$ be a matrix compact set in V. Then for each level X_n, we can find a sequence of points in $x_{n,i} \in M_n(V)$ which converge to zero, such that

$$X_n \subseteq \overline{\mathrm{co}(\{x_{n,1}, \ldots, x_{n,m}, \ldots\})}.$$

Let x_n be the matrix in $M_n(V) \check{\otimes}_{\mathrm{op}} \mathcal{B}(\ell^2) \cong V \check{\otimes}_{\mathrm{op}} \mathcal{B}(\ell^2)$ given by

$$x_n = \mathrm{diag}(x_{n,1}, \ldots, x_{n,m}, \ldots)$$

and let x be the element of $V \check{\otimes}_{\mathrm{op}} \mathcal{B}(\ell^2) \check{\otimes}_{\mathrm{op}} \mathcal{B}(\ell^2)$ given by

$$x = \mathrm{diag}(x_1, \ldots, x_n, \ldots).$$

Now we observe that any element $v \in \mathrm{co}(\{x_{n,1}, \ldots, x_{n,m}, \ldots\})$ can be written as $(\mathrm{id} \otimes \sigma)(x)$ for some $\sigma \in \mathcal{CB}(\mathcal{B}(\ell^2) \check{\otimes}_{\mathrm{op}} \mathcal{B}(\ell^2), M_n)$, with $||\sigma||_{\mathcal{CB}} \leq 1$. Now since $\varphi_\nu \to \varphi$ in the strongly stable point-norm topology, we have that this implies by [9] that

$$\varphi_\nu \otimes \mathrm{id} \to \varphi \otimes \mathrm{id}$$

point-norm in $\mathcal{CB}(V \check{\otimes}_{\mathrm{op}} Z, W \check{\otimes}_{\mathrm{op}})$, so in particular, we can find an N such that

$$||(\varphi_\nu \otimes \mathrm{id})(x) - (\varphi \otimes \mathrm{id})(x)|| < \varepsilon$$

for all $\nu \geq N$, and so

$$\begin{aligned} \|(\varphi_\nu)_n(v) - \varphi_n(v)\| &= \|(\varphi_\nu \otimes \mathrm{id})(v) - (\varphi \otimes \mathrm{id})(v)\| \\ &= \|(\mathrm{id} \otimes \sigma)(\varphi_\nu \otimes \mathrm{id})(x) - (\varphi \otimes \mathrm{id})(x)\| \\ &< \varepsilon \end{aligned}$$

for all $\nu \geq N$, and for any $n \in \mathbb{N}$ and $v \in \mathrm{co}(\{x_{n,1}, \ldots, x_{n,m}, \ldots\})$. So φ_ν converges to φ completely uniformly on the matrix set

$$\{\mathrm{co}(\{x_{n,1}, \ldots, x_{n,m}, \ldots\})\},$$

and so a simple $\varepsilon/3$ argument gives us convergence on $\boldsymbol{X}$. □

So we have proved that these three forms of convergence are equivalent. This means that any of these three can be substituted for the type of convergence in the strong operator approximation property. Kirchberg's result then tells us that there is an operator space where the strongly stable point-norm topology is different from the stable point-norm topology. Hence by the above result and Proposition 4.3.1 the topology of completely uniform convergence on strongly operator compact sets does not agree with the topology of completely uniform convergence on operator compact sets. This implies:

Lemma 4.4.3
There is a strongly operator compact set $\boldsymbol{X}$ in some operator space V such that $\boldsymbol{X}$ is not operator compact.

Proof :
Let V be a space where the operator approximation property holds, but not the strong operator approximation property. Assume that there was not such $\boldsymbol{X}$ in this V. Then the topology of completely uniform convergence on strongly operator compact sets agrees with the topology of completely uniform convergence on operator compact sets, as there is no difference in the classes of matrix sets, and so we have a contradiction by our previous discussion. □

Turning this around we can give a condition for when the operator approximation property will imply the strong operator approximation property.

Proposition 4.4.4
If an operator space V satisfies the operator approximation property and every strongly operator compact set is operator compact, then V satisfies the strong operator approximation property.

Corollary 4.4.5
If a C^*-algebra A satisfies the operator approximation property and every strongly operator compact set is operator compact, then A is exact.

This may not be a complete characterization, however, since it is conceivable that the operator and strongly operator compact sets may be different, but the topologies of completely uniform convergence agree.

To conclude this section, we will note that there are a couple of partial results which are analogous to Theorem 4.3.7.

Proposition 4.4.6
If an operator space V has the strong operator approximation property, then for any operator space W and any strongly operator compact (resp. matrix compact) map $\varphi \in \mathcal{CB}(W, V)$, φ can be approximated completely uniformly by finite rank maps.

Proof :
There exists a strongly operator compact (resp. matrix compact) set $\boldsymbol{X}$ in V such that $\boldsymbol{\varphi}(\boldsymbol{B}^W) \subseteq \boldsymbol{X}$.

Since V has the strong approximation property, for any $\varepsilon > 0$ there is a finite rank map $\psi \in \mathcal{CB}(V, V)$ such that

$$\|\psi(v) - v\| < \varepsilon$$

for all $v \in \boldsymbol{X}$ and so

$$\|\psi \circ \varphi - \varphi\|_{cb} < \varepsilon.$$

□

4.5 Co-exactness

Given Proposition 4.4.4 and its corollaries, it would useful to be able to say exactly when strong operator compactness implies operator compactness. The following lemma starts us off in the direction we want to go.

Lemma 4.5.1
Let $V = \mathcal{T}_n/W$, that is V is a finite quotient of the n by n trace class operators, then the matrix unit ball of V is operator compact.

Proof :
By Example 4.1.1, we know that the unit ball of $\mathcal{T}_n$ lies inside $\mathbf{co}(\tau)$. So if v is in the mth level of the matrix unit ball of V, then there is some σ in the mth level of the matrix unit ball of $\mathcal{T}_n$ such that $v = \pi_m(\sigma)$, where π is the quotient map. Then

$$v = (\mathrm{id} \otimes \pi)(\sigma \otimes \mathrm{id})(\sum_{i,j=1}^{n} e_{i,j} \otimes \tau_{i,j}) = (\sigma \otimes \mathrm{id})(\sum_{i,j=1}^{n} e_{i,j} \otimes \pi(\tau_{i,j}))$$

so the unit ball lies in $\mathbf{co}\, \pi_n(\tau)$. □

Corollary 4.5.2
If V is any finite dimensional operator space, then the matrix unit ball of V is operator compact.

Proof :
For V is completely isomorphic (but maybe not completely isometric) via φ to $\mathcal{T}_n/W$ for some n and some W. Without loss of generality we may take $\|\varphi\|_{cb} = 1$. Hence if we let $x = \varphi_n^{-1}(\pi_n(\tau))$, we have that the matrix unit ball $\boldsymbol{X}$ of V has image $\boldsymbol{\varphi}(\boldsymbol{X})$ inside the hull of $\pi_n(\tau)$ and so $\boldsymbol{X} \subseteq \mathbf{co}(x)$. □

We notice that the x in the corollary may have arbitrarily large norm. It will be important for us to be able to measure how big the norm of x can get. To do this we recall [7] that the completely bounded Banach-Mazur distance, introduced by Pisier [32], between two finite dimensional operator spaces V and W is defined to be

$$d_{cb}(V, W) = \inf\{\|\varphi\|_{cb}\|\varphi^{-1}\|_{cb} : \varphi \in \mathcal{CB}(V, W) \text{ a complete isomorphism}\}.$$

Recall [7, 32] that a finite dimensional operator space V is λ-*exact* if for any $\varepsilon > 0$, we can find a subspace W of M_n such that

$$d_{cb}(V, W) < \lambda + \varepsilon.$$

An infinite dimensional operator space is λ-exact if every finite dimensional subspace is λ-exact. We define $d_{ex}(V)$ to be the infimum of the λ for which V is λ-exact an ∞ if there is no such λ.

What we want to measure is a sort of dual of this notion: we are interested in the distance from quotients of $\mathcal{T}_n$, so we will say that V is λ-*coexact* if for every $\varepsilon > 0$, we can find a quotient W of $\mathcal{T}_n$ such that

$$d_{cb}(V, W) < \lambda + \varepsilon.$$

If V is infinite dimensional, we say V is λ-coexact if every finite dimensional subspace is λ-coexact. We define $d_{cex}(V)$ to be the infimum of the λ for which V is λ-coexact an ∞ if there is no such λ.

We note that if V is finite dimensional with $d_{cex}(V) = \lambda$, then for any $\varepsilon > 0$, we can find an x such that the unit ball of V lies in $\mathbf{co}(x)$ such that $\|x\| \leq \lambda + \varepsilon$. So coexactness measures how much we have to inflate τ to get x.

To see how this condition helps us we will consider the way that one shows that compact sets in a Banach space V are subsets of the closed convex hulls of sequences which converge to zero. Let K be our compact set. Choose a finite dimensional V_1 so that for every $x \in 2K$, there is a $v \in V_1$ so that

$$\|x - v\| \leq 1/4.$$

Now we can find a polygon in V_1 with vertices $x_{1,1}, \ldots, x_{1,n_1}$ so that

$$\{v \in V_1 : d(v, 2K) < 1/4\} \subseteq \operatorname{co}(\{x_{1,1}, \ldots, x_{1,n_1}\})$$

and

$$\|(x_{1,1}, \ldots, x_{1,n_1})\|_\infty \leq \sup\{\|x\| : x \in 2K\} + 1/4 + \varepsilon_1.$$

This last claim follows because for any $\varepsilon > 0$, we can find a finite dimensional W which is a subspace of ℓ_n^∞ for some n, so that

$$d_b(W, V_1^*) < 1 + \varepsilon$$

where d_b is the classical Banach-Mazur distance. Then if $\varphi : W \to V_1^*$ is an isomorphism such that $\|\varphi\|\|\varphi^{-1}\| < 1 + \varepsilon_1$, we have that $\varphi^* : V_1 \to W^*$ is an isomorphism which satisfies $\|\varphi^*\|\|(\varphi^*)^{-1}\| < 1 + \varepsilon_1$, and W^* is a quotient of ℓ_n^1. In other words *every finite dimensional Banach space satisfies the classical version of 1-coexactness*. We then let x_i be the image of $e_i \in \ell_n^1$ under the quotient mapping and $(\varphi^*)^{-1}$.

Now we let $K_2 = 2K + V_1 \subseteq V/V_1$. K_2 is compact, so we can repeat the process to get a V_2 within $1/16$ of $2K_2$ and $x'_{2,1}, \ldots, x'_{2,n_2} \in V_1$ which satisfy the analogous conditions (but with a ε'_2). We know that we can find $x_{2,1}, \ldots, x_{2,n_2} \in V$ such that $\pi_1(x_{2,i}) = x'_{2,i}$ (where π_1 is the projection map $V \to V/V_1$) and so that

$$\|(x_{2,1}, \ldots, x_{2,n_2})\| - \|(x'_{2,1}, \ldots, x'_{2,n_2})\| < \varepsilon_2.$$

We repeat this construction. We let $x = (x_{1,1}, \ldots, x_{1,n_1}, x_{2,1}, \ldots, x_{2,n_2}, \ldots)$ we note that for $k \geq 2$

$$\sup\{x_{k,i}\}\| \leq 2^{-k} + 4^{-k} + 2\varepsilon_k$$

so that if we choose ε_k correctly, then $x \in c_0 \check{\otimes} V$.

Moreover, let $v \in K$, we can find $v_1 \in \operatorname{co}(x_1)$, so that

$$\|2v - v_1\| \leq 1/4 + \varepsilon_1,$$

and $v_2 \in \operatorname{co}(x_2)$ such that

$$\|2(2v - v_1) - v_2\| \leq 1/16 + 2\varepsilon_2$$

and so on so that in general we have $v_1, \ldots, v_n$ in $\operatorname{co}(x_1), \ldots, \operatorname{co}(x_n)$ respectively, so that

$$\|2^n v - 2^{n-1}v_1 - \cdots - v_n\| \leq \frac{1}{4^n} + 2\varepsilon_n$$

and so

$$\|v - (2^{-1}v_1 + \cdots + 2^{-n}v_n)\| \leq \varepsilon.$$

But $2^{-1}v_1 + \cdots + 2^{-n}v_n \in \operatorname{co}(x)$, and so $K \subseteq \overline{\operatorname{co}(x)}$.

The key fact which allowed this construction to work was the observation about classical coexactness. Which leads us to the following.

Theorem 4.5.3
Let V be a λ-coexact operator space for $\lambda < \infty$. If $\boldsymbol{K}$ is a strongly operator compact subset of $\boldsymbol{V}$, then $\boldsymbol{K}$ is operator compact.

Proof :
The proof is essentially the operator space version of the preceding argument.
Choose a finite dimensional V_1 so that for every matrix point $\boldsymbol{x} \in 2\boldsymbol{K}$, there is a matrix point $\boldsymbol{v} \in \boldsymbol{V_1}$ so that

$$\|\boldsymbol{x} - \boldsymbol{v}\| \leq 1/4.$$

Now we can find an $x_1 \in M_{n_1}(V_1)$ so that

$$\{v \in V_1 : d(v, 2K) < 1/4\} \subseteq \mathbf{co}(x_1)$$

and

$$\|x_1\| \leq \lambda \sup\{\|x\| : x \in 2K\} + 1/4 + \varepsilon_1.$$

This last claim follows because for any $\varepsilon > 0$, we can find a finite dimensional W which is a subspace of ℓ_n^∞ for some n, so that

$$d_b(W, V_1^*) < \lambda + \varepsilon.$$

Now we let $\boldsymbol{K_2} = 2\boldsymbol{K} + \boldsymbol{V_1} \subseteq \boldsymbol{V}/\boldsymbol{V_1}$. $\boldsymbol{K_2}$ is strongly operator compact, since if we approximate $\boldsymbol{K}$ within ε by V_ε, then $(V_\varepsilon \cap V_1)/V_1$ will approximate $\boldsymbol{K_2}$ within ε. So we can repeat the process to get a V_2 within $1/16$ of $2\boldsymbol{K_2}$ and $x_2' \in M_{n_2}(V_1)$ which satisfy the analogous conditions (but with a ε_2'). We know that we can find $x_2 \in M_{n_2}(V)$ such that $\pi_1(x_2) = x_2'$ (where π_1 is the projection map $V \to V/V_1$) and so that

$$\|x_2\| - \|x_2'\| < \varepsilon_2.$$

We repeat this construction. We let $x = \operatorname{diag}(x_1, x_2, \dots)$ we note that for $k \geq 2$

$$\sup\{x_k\}\| \leq \lambda 2^{-k} + 4^{-k} + 2\varepsilon_k$$

so that if we choose ε_k correctly, then $x \in \mathcal{K} \check{\otimes}_{\mathrm{op}} V$.
Moreover, let $v \in \boldsymbol{K}$, we can find $v_1 \in \mathbf{co}(x_1)$, so that

$$\|2v - v_1\| \leq 1/4 + \varepsilon_1,$$

and $v_2 \in \mathbf{co}(x_2)$ such that

$$\|2(2v - v_1) - v_2\| \leq 1/16 + 2\varepsilon_2$$

and so on so that in general we have $v_1, \dots, v_n$ in $\mathbf{co}(x_1), \dots, \mathbf{co}(x_n)$ respectively, so that

$$\|2^n v - 2^{n-1} v_1 - \cdots - v_n\| \leq \frac{1}{4^n} + 2\varepsilon_n$$

and so

$$\|v - (2^{-1}v_1 + \cdots + 2^{-n}v_n)\| \leq \varepsilon.$$

But $2^{-1}v_1 + \cdots + 2^{-n}v_n \in \mathbf{co}(x)$, and so $K \subseteq \overline{\mathbf{co}(x)}$.
Hence $\boldsymbol{K}$ is operator compact. □

Corollary 4.5.4
If V is a λ-coexact (where $\lambda < \infty$) operator space and satisfies the operator approximation property, then V satisfies the strong operator approximation property.

Corollary 4.5.5
If A is a λ-coexact (where $\lambda < \infty$) C^*-algebra and satisfies the operator approximation property, then A is exact.

Corollary 4.5.6
There are operator spaces which are not λ-coexact for any $\lambda < \infty$.

One might ask what other properties that coexact spaces have, and as a result in that direction, we show that 1-coexactness is related to locally reflexivity.

Proposition 4.5.7
If V is an operator space and V^{**} is 1-coexact, then V is locally reflexive.

Proof :
We want to show that if $\varphi : W \hookrightarrow V^{**}$, then φ can be approximated in the weak-* topology by complete contractions $\varphi_\nu \in \mathcal{CB}(W, V)$.

We first assume that $W = \mathcal{T}_n/X$ for some X, so $W^* = X^\perp \subseteq M_n$, and we know that $M_n(V^*) = \mathcal{T}_n(V)^*$, so that

$$\begin{array}{ccc} (M_n \check{\otimes}_{\mathrm{op}} V)^* & \cong & \mathcal{T}_n \hat{\otimes}_{\mathrm{op}} V^* \\ \downarrow & & \downarrow \\ (X^\perp \check{\otimes}_{\mathrm{op}} V)^* & \cong & W \hat{\otimes}_{\mathrm{op}} V^* \end{array}$$

and the bottom row is a complete isometry. Hence

$$\mathcal{CB}(W, V^{**}) \cong (W \hat{\otimes}_{\mathrm{op}} V^*)^* \cong (X^\perp \check{\otimes}_{\mathrm{op}} V)^{**} \cong \mathcal{CB}(W, V)^{**}$$

and by the matrix bipolar theorem, we know that the unit ball of $\mathcal{CB}(W, V)$ is weakly dense in the unit ball of $\mathcal{CB}(W, V)^{**}$, so that there is a net of complete contractions $\varphi_\nu \in \mathcal{CB}(W, V)$ which converges to φ, which means that for all $x \in \boldsymbol{W}$ and $\psi \in \boldsymbol{V^*}$, we have

$$\langle\langle \psi, \boldsymbol{\varphi_\nu}(x) \rangle\rangle = \langle\langle x \otimes \psi, \varphi_\nu \rangle\rangle \to \langle\langle x \otimes \psi, \varphi \rangle\rangle = \langle\langle \psi, \boldsymbol{\varphi}(x) \rangle\rangle.$$

So now if V is 1-coexact, for any $\varepsilon > 0$, we can find a complete isomorphism $\theta : W \to \mathcal{T}_n/X$ for some n and for some X, such that $||\theta||_{cb}||\theta^{-1}||_{cb} \leq 1 + \varepsilon$. So if we let $||\theta^{-1}|| = 1$, we have that

$$\psi = \varphi \circ \theta^{-1} : \mathcal{T}_n/X \to V^{**}$$

can be approximated by $\psi_\nu : \mathcal{T}_n/X \to V$, but if we let

$$\varphi_\nu = \psi_\nu \circ \theta : W \to V$$

we have that $\varphi_\nu \to \varphi$ weak-*, and $||\varphi_\nu|| \leq 1 + \varepsilon$. By normalizing the φ_ν to get complete contractions and then repeating as $\varepsilon \to 0$, we get complete contractions approximating φ weak-*. $\square$

References

[1] W. Arveson, *Subalgebras of C*-algebras*, Acta Math. **123** (1969), 141–224.

[2] D. P. Blecher, *The standard dual of an operator space*, Pacific J. Math. **153** (1992), 15–30.

[3] D. P. Blecher and V. I. Paulsen, *Tensor products of operator spaces*, J. Funct. Anal. **99** (1991), 262–292.

[4] N. Bourbaki, *Topological vector spaces*, Elements of Mathematics, Springer-Verlag, New York, 1987.

[5] A. Connes, *Noncommuntative geometry*, Academic Press, New York, 1994.

[6] D. A. Dubin and M. A. Hennings, *Quantum mechanics, algebras and distributions*, Longman Scientific & Technical, Essex, 1990.

[7] E. G. Effros and Z.-J. Ruan, *Operator spaces*, Oxford University Press, to appear.

[8] ———, *On matricially normed spaces*, Pacific J. Math. **132** (1988), no. 2, 243–264.

[9] ———, *On approximation properties for operator spaces*, Int. J. Math. **1** (1990), no. 2, 162–187.

[10] ———, *Self-duality for the Haagerup tensor product and Hilbert space factorizations*, J. Funct. Anal. **200** (1991), 257–284.

[11] ———, *On the abstract characterization of operator spaces*, Proc. Amer. Math. Soc. **119** (1993), 579–584.

[12] ———, *The Grothendieck-Pietsch and Dvoretsky-Rogers theorems for operator spaces*, J. Funct. Anal. **122** (1994), no. 2, 428–450.

[13] ———, *Mapping spaces and liftings for operator spaces*, Proc. London Math. Soc. (3) **69** (1994), 171–194.

[14] E. G. Effros and C. Webster, *Operator analogues of locally convex spaces*, to appear, Proceedings of the Aegean Conference on Operator Algebras and Applications, 1996.

[15] E. G. Effros and S. Winkler, *Matrix convexity: Operator analogues of the bipolar and Hahn-Banach theorems.*, J. Funct. Anal. **144** (1995), no. 1, 117–152.

[16] D. R. Farenick and P. B. Morenz, *C*-extreme points of some compact convex sets*, Proc. Amer. Math. Soc. **118** (1993), 765–775.

[17] P. A. Fillmore, *A user's guide to operator algebras*, Wiley Interscience, New York, 1996.

[18] Gelfand and Naimark, *Generalized functions*, Academic Press, 1950.

[19] A. Hopenwasser, R. L. Moore, and V. I. Paulsen, *C*-extreme points*, Trans. Amer. Math. Soc. **266** (1981), 291–307.

[20] B. E. Johnson, *C^*-operator spaces*, unpublished notes.

[21] Kadison and Ringrose, *Fundamentals of the theory of operator algebras*, Academic Press, New York, 1983.

[22] E. Kirchberg, *On the matricial approximation property*, preprint, 1991.

[23] ______, *On non-semisplit extensions, tensor products and exactness of group C^*-algebras*, Invent. Math. **112** (1993), no. 3, 449–489.

[24] J. Lindenstrauss and L. Tzafriri, *Classical Banach spaces I and II*, Springer, New York, 1996.

[25] R. Loebl and V. I. Paulsen, *Some remarks on C^* convexity*, Linear Alg. and Appl. **78** (1981), 63–78.

[26] J. Murray and J. von Neumann, *On rings of operators, IV*, Ann. Math. **44** (1943), 716–808.

[27] V. I. Paulsen, *Completely bounded maps and dilations*, Notes in Mathematics, no. 146, Pitman, New York, 1986.

[28] G. Pedersen, *C^*-algebras and their automorphism groups*, Academic Press, London, 1979.

[29] N. C. Phillips, *Inverse limits of C^*-algebras*, J. Operator Theory **19** (1988), 159–195.

[30] G. Pisier, *An introduction to the theory of operator spaces*, preprint.

[31] ______, *Espaces de Banach quantiques: Une introduction à la théorie des espaces d'operatuers*, Journee Annuelle Soc. Math. France (1994), 1–59.

[32] ______, *Exact operator spaces*, Asterisque (1995), no. 232, 159–186.

[33] A. P. Robertson and W. J. Robertson, *Topological vector spaces*, Cambridge University Press, London, 1964.

[34] Z.-J. Ruan, *Subspaces of C^*-algebras*, J. Funct. Anal. **76** (1988), 217–230.

[35] H. Saar, *Kompacte, vollständig beschränkte abbilungen mit werten in einer nuklearen C^*-algebra*, Diplomarbeit, Universität des Saarlandes, 1982.

[36] K. Schmüdgen, *Unbounded operator algebras and representation theory*, Operator Theory: Advances and Applications, vol. 37, Birkhäuser, Boston, 1990.

[37] M. Takesaki, *Theory of operator algebras, I*, Springer-Verlag, New York, 1979.

[38] F. Treves, *Topological vector spaces, distributions and kernels*, Academic Press, New York, 1967.

[39] D. Voiculescu, K. Dykema, and A. Nica, *Free random variables*, American Mathematical Society, 1992.

[40] N. Weaver, *Operator spaces and noncommutative metrics*, preprint, 1995.

[41] C. Webster and S. Winkler, *The Krein-Milman theorem in operator convexity*, preprint, 1997.

[42] S. Winkler, *Matrix convexity*, Ph.D. thesis, University of California, Los Angeles, 1996.

[43] G. Wittstock, *Ein operatwertiger Hahn-Banach satz*, J. Funct. Anal. **40** (1981), 127–150.

[44] Y.-C. Wong, *Schwartz spaces, nuclear spaces and tensor products*, Lecture Notes in Mathematics, no. 726, Springer-Verlag, New York, 1979.