# OPERATOR ANALOGUES OF LOCALLY CONVEX SPACES

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*Dedicated to the memory of Lajos Pukanszky.*

ABSTRACT. Local operator spaces are defined to be projective limits of operator spaces. These limits arise when one considers linear spaces of unbounded operators, and they may be regarded as the "quantized" or "operator" analogues of locally convex spaces. It is shown that for nuclear spaces, the maximal and minimal quantizations coincide. Thus in a striking contrast to normed spaces, a nuclear space has precisely one quantization. Furthermore, it is shown that a local operator space is nuclear in the operator sense if and only if its underlying locally convex space is nuclear. Operator versions of bornology and duality are also considered.

## 1. INTRODUCTION

Quantum physics has provided functional analysts with some of their most challenging problems. In order to justify the physical theory, it is necessary to consider linear spaces of (generally unbounded) Hilbert space operators. Although in some cases, such as the theory of "functional integrals" on nuclear spaces, functional analytic methods have proved to be remarkably effective (see [14], [26]), a generally accepted framework for the Feynman integral remains elusive.

In this paper we consider an aspect of the functional analytic theory that may have a bearing on these problems. The transition from functions to operators introduces a new kind of linear structure into the systems. A linear space of operators is automatically equipped with natural orderings or norms on its matrices, which is usually not determined by those structures on the space itself. This phenomenon was first observed by operator algebraists, who found it necessary to replace the positive and bounded mappings of classical analysis with those having these properties on the matrices as well (see, e.g., [1]).

In recent years, several models have been introduced in order to better understand these notions. M.-D. Choi and the first author axiomatized self-adjoint unital linear spaces of *bounded* operators [3] by considering the underlying matrix orderings. A more inclusive theory was formulated by Z.-J. Ruan, who succeeded in characterizing arbitrary linear spaces of bounded operators by using the intrinsic matrix norms [24]. These spaces are known as *operator spaces,* and they may be thought of as "quantized normed spaces". There is now a rapidly growing literature on the theory and applications of these objects.

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R. Powers was the first to consider the matrix structure underlying spaces of unbounded operators. Focussing his attention on the enveloping algebra $\mathcal{A}(\mathfrak{g})$ of a Lie algebra $\mathfrak{g}$, he showed that one can identify the integrable representations of $\mathfrak{g}$ by using suitable *matrix orderings* on $\mathcal{A}(\mathfrak{g})$ [22]. In this paper we consider a non-ordered theory that is applicable to more general linear spaces. Since locally convex spaces are just the projective limits of normed spaces (see [23]), it is natural to define a *local operator space* to be a projective limit of operator spaces. We wish to show that this simple idea leads to a satisfactory theory.

We have followed the general program proposed some forty years ago by Grothendieck for the study of locally convex vector spaces (see [16], [17], [15], and especially [20]). In that pioneering work, Grothendieck carefully examined the relationship between various tensor products and mapping spaces. His approach has thus far proved to be the most effective way of generalizing classical Banach space techniques to operator space theory (see, e.g., [2], [10], [7], [19], [21]).

The theory of locally convex spaces is *purely topological,* i.e., it is concerned with the topology on a vector space, rather than any specific seminorms. To illustrate this, we recall that any two locally convex spaces of the same finite dimension are linearly homeomorphic, and thus may be identified as locally convex spaces. Similarly, a local operator space $V$ is characterized by a *matrix topology* on the linear space $\mathfrak{m}(V)$ of infinite matrices $[v_{i,j}]_{i,j\in\mathbb{N}}$ with $v_{i,j} \in V$ zero for all but finitely many indices. Once again, a linear isomorphism between two finite dimensional local operator spaces induces a linear homeomorphism of their matrix spaces (see §6). On the other hand, an infinite dimensional Hilbert space $H$ supports non-completely isomorphic operator space structures, and the corresponding topologies on $\mathfrak{m}(H)$ are therefore distinct (see §5 for more details).

It should also be noted that the matrix topology on $\mathfrak{m}(V)$ for $V$ a local operator space is *asymptotic*, i.e., it cannot be detected on matrices of bounded size. Letting $V$ and the $n \times n$ matrices $M_n(V)$ have the relative topologies in $\mathfrak{m}(V)$, we have the natural linear homeomorphism

$$M_n(V) \cong V^{n^2}, \tag{1}$$

where we have used the product topology on the right (see §5).

We begin in §2 and §3 with a discussion of the notions of absolute matrix convexity for sets and their gauges. In §4 we prove a version of the bipolar theorem. The local operator spaces are introduced in §5. In §6 we define the local operator space versions of the projective and injective tensor products by using projective limits of the corresponding operator space products.

We use the operator projective tensor product to define the notion of matrix nuclearity for local operator spaces in §7. We prove that a local operator space is nuclear if and only if the underlying locally convex space is nuclear. It will be recalled that the nuclear spaces behave in many respects like finite dimensional spaces. This notion is reinforced by the surprising fact that *a nuclear locally convex space has precisely one quantization.* We consider the matrix analogues of boundedness and "bornology" in §8, and we then discuss local matrix topologies for mapping spaces and duals in §9. We conclude in §10 with a simple example of a nuclear operator space that is relevant to elementary quantum mechanics.

Local operator spaces display unusual non-classical phenomena that require additional care. The appropriate bipolar theorem involves a few subtleties, which have already been considered in [13]. Owing to the fact that operator spaces need not be locally reflexive in the appropriate sense (this is in contrast to the situation for normed spaces) [5], the theory of operator integral operators is also more difficult (see §7).

We have chosen not to be encyclopedic. Instead, our goal has been to prove a selection of results that show how one can also use Grothendieck's theory in the setting of local operator spaces. We will explore other aspects of the theory, such as the Haagerup tensor product (see §6), in a subsequent paper.

We are indebted to Marc Rieffel, who for some time has encouraged us to develop the local theory. His studies of quantum deformations had led him as well to conclude that projective limits of operator spaces will play an important role in non-commutative analysis. We also wish to thank Barry Johnson, who has advocated a Banach module approach to operator space theory. His influence may be found in our frequent use of matrix modules, and in particular, in his elegant characterization of $\mathfrak{m}$-convex hulls in Lemma 3.2.

Turning to notation, we define a *graded set* $S_{\circledast} = (S_n)$ to be a sequence of sets $S_n$ $(n \in \mathbb{N})$. Given a sequence of points $x_{\circledast} = (x_n)$, we write $x_{\circledast} \in S_{\circledast}$ if $x_n \in S_n$ for all $n$. Given two graded sets $S_{\circledast}$ and $T_{\circledast}$, we write $S_{\circledast} \subseteq T_{\circledast}$ if $S_n \subseteq T_n$ for all $n \in \mathbb{N}$. When convenient, we omit the symbol $\circledast$.

We let $l_\infty, l_2, l_1$, and $c_0$ denote the usual sequence spaces, and $M_\infty, HS_\infty, T_\infty$, and $K_\infty$ denote the operator analogues, i.e., the Banach spaces of bounded operators, Hilbert-Schmidt, trace class, and compact operators, respectively.

In this paper all vector spaces are assumed complex. Given vector spaces $V$ and $W$, we let $L(V,W)$ denote the vector space of linear mappings $\varphi : V \to W$, and we let $V^d = L(V,\mathbb{C})$. We define a *pairing* of complex vector spaces $V$ and $W$ to be a bilinear function $F : V \times W \to \mathbb{C}$ for which the corresponding functions $V \to W^d$ and $W \to V^d$ are injective. If we have fixed $F$, we will often use the notation $\langle v, w\rangle = F(v,w)$, whereas we will use the notation $\langle\cdot|\cdot\rangle$ for sesquilinear forms.

All locally convex spaces in this paper are assumed to be Hausdorff. We have found it convenient to use slightly different notations for normed and locally convex spaces. Given normed spaces, we let $\mathcal{B}(V,W)$ denote the normed space of bounded linear mappings from $V$ to $W$, and $V^* = \mathcal{B}(V,\mathbb{C})$. For locally convex spaces we use the notation $\mathcal{C}(V,W)$ for the vector space of continuous linear mappings $\varphi : V \to W$, and $V' = \mathcal{C}(V,\mathbb{C})$. On the other hand we let $\mathcal{B}(V,W)$ denote the bounded linear mappings $\varphi : V \to W$ (these need not be continuous – see §9).

The reader may read about operator spaces in [2], [11], and [12]. We use the following slightly modified notation. If $V$ and $W$ are operator spaces, we let $\mathcal{B}_{op}(V,W)$ denote the completely bounded mappings from $V$ to $W$ with the completely bounded operator norm $\|\cdot\|_{cb}$. This may be regarded as an operator space structure provided one uses the usual identification

$$M_n(\mathcal{B}_{op}(V,W)) \cong \mathcal{B}_{op}(V, M_n(W)).$$

On the other hand, we may identify the normed spaces $\mathcal{B}_{op}(V,\mathbb{C})$ and $V^*$, and the operator space structure on the latter space is determined by the linear isomorphism

$$M_n(V^*) \cong \mathcal{B}_{op}(V, M_n). \tag{2}$$

Finally we recall that there are two especially important operator structures that can be placed on a Hilbert space $H$. We may use the linear isomorphisms

$$H \cong \mathcal{B}(\mathbb{C}, H)$$

and

$$H \cong \mathcal{B}(H^*, \mathbb{C})$$

of $H$ with "column" and "row" operators to determine the column and row matrix norms $\rho_c$ and $\rho_r$ on $H$, and we denote the corresponding *column* and *row* Hilbert operator spaces by $H_c$ and $H_r$, respectively (see, e.g., [8]). To be more explicit, the corresponding matrix norms are determined by the linear identifications

$$M_{m,n}(H_c) = \mathcal{B}(\mathbb{C}^n, H^m),$$

and

$$M_{m,n}(H_r) = \mathcal{B}(H^{*n}, \mathbb{C}^m).$$

We have natural complete isometries $(H_c)^* \cong (H^*)_r$.

There is a basic terminological difficulty with matrix structures, since the terms "completely", "operator", and "matrix" have all been used to signal that one is considering a property of the order or topology on $\mathfrak{m}(V)$ ratherthan on $V$. Unfortunately none of these terms would be satisfactory in all situations since, for example, the term "completely continuous" would conflict with an unrelated notion. We have therefore chosen to use these terms interchangeably. It should also be noted that the "operator" or "matrix" nuclear spaces considered below have nothing to do with nuclear $C^*$-algebras. Since the theory of nuclear locally convex spaces preceded that of nuclear $C^*$-algebras by several decades, we propose that the latter be called $C^*$-nuclear if there is a possibility of confusion.

## 2. Matrix conventions

We let $M_{m,n}$ denote the vector space of complex $m$ by $n$ matrices $\alpha = [\alpha_{i,j}]$, and $M_n = M_{n,n}$. We write $\varepsilon_{i,j}^{m,n}$ $(1 \leq i \leq m, 1 \leq j \leq n)$ for the usual basis of matrix units in $M_{m,n}$, and we let $\varepsilon_{i,j}^n = \varepsilon_{i,j}^{n,n}$. We identify $M_{m,n}$ with the normed space $\mathcal{B}(\mathbb{C}^n, \mathbb{C}^m)$ and we regard the conjugate transpose operation $\alpha \mapsto \alpha^*$ as an isometric conjugate linear mapping from $M_{m,n}$ to $M_{n,m}$. Identifying $M_n$ with the $C^*$-algebra $\mathcal{B}(\mathbb{C}^n)$, $M_n$ has the multiplicative identity $\varepsilon^n = \sum \varepsilon_{i,i}^n$.

Given a (complex) vector space $V$, we let $M_{m,n}(V)$ be the vector space of all $m$ by $n$ matrices $v = [v_{i,j}]$ with $v_{i,j} \in V$, and we let $M_n(V) = M_{n,n}(V)$, and $M_{m,n} = M_{m,n}(\mathbb{C})$. We identify a matrix of matrices over $V$ with a single large matrix by deleting the inner brackets. Given $v \in M_{m,n}(V)$ and $w \in M_{p,q}(V)$, and scalar matrices $\alpha \in M_{p,m}$ and $\beta \in M_{n,q}$, we define

$$v \oplus w \in M_{m+p,n+q}(V)$$

and

$$\alpha v \beta \in M_{p,q}(V)$$

in the obvious manner. Given $v = [v_{i,j}] \in M_{p,q}(V)$, we have that

$$v_{i,j} = E_i^{(p)} v E_j^{(q)*}, \tag{3}$$

where in general we let

$$E_k^{(n)} = [0 \dots 1_k \dots 0_n] \in M_{1,n}.$$

On the other hand we have that

$$v = \sum_{i,j} E_i^{(p)*} v_{i,j} E_j^{(q)} \tag{4}$$

In many situations it is advantageous to replace the finite matrix spaces $M_{m,n}(V)$ with the space $\mathfrak{m}(V)$ of infinite matrices $[v_{i,j}]$ ($v_{i,j} \in V, i,j \in \mathbb{N}$), where all but finitely many of the $v_{i,j}$ are zero. Unless we indicate otherwise, we use the mapping

$$\begin{bmatrix} v_{1,1} & \dots & v_{1,n} \\ & & \\ v_{m,1} & \dots & v_{m,n} \end{bmatrix} \mapsto \begin{bmatrix} v_{1,1} & \dots & v_{1,n} & 0 & \dots \\ & & & & \\ v_{m,1} & \dots & v_{m,n} & 0 & \dots \\ 0 & \dots & 0 & 0 & \\ \dots & & \dots & & \dots \end{bmatrix}$$

to identify $M_{m,n}(V)$ with a subspace of $\mathfrak{m}(V)$. Letting $\mathfrak{m} = \mathfrak{m}(\mathbb{C})$, we have that $\mathfrak{m}(V)$ is an $\mathfrak{m}$-bimodule in the obvious manner. It will also be useful to consider more general index sets. If $\mathfrak{s}$ and $\mathfrak{t}$ are finite sets, we let $M_{\mathfrak{s},\mathfrak{t}}(V)$ be the vector space of matrices $[v_{i,j}]$ ($i \in \mathfrak{s}, j \in \mathfrak{t}$), and given arbitrary sets $\mathfrak{s}$ and $\mathfrak{t}$, we define

$$\mathfrak{m}_{\mathfrak{s},t}(V) = \lim_{\rightarrow} M_{\mathfrak{s}_0,\mathfrak{t}_0}(V),$$

where the limit is taken over finite subsets $\mathfrak{s}_0 \subseteq \mathfrak{s}$, and $\mathfrak{t}_0 \subseteq \mathfrak{t}$. In this more general context it is useful to let $n \in \mathbb{N}$ also stand for the set $\{1, \dots, n\}$, and $\infty$ for the set $\mathbb{N}$. Our previous conventions have obvious formulations in this more general setting. Letting $\varepsilon_{i,j} \otimes \varepsilon_{k,l}$ correspond to $\varepsilon_{(i,k),(j,l)}$, we have a natural identification

$$\mathfrak{m} \otimes \mathfrak{m} \cong \mathfrak{m}_{\infty \times \infty}.$$

Each linear mapping $\varphi : V \to W$ determines linear mappings

$$\varphi_n : M_n(V) \to M_n(W) : [v_{i,j}] \mapsto [\varphi(v_{i,j})]$$

and

$$\varphi_\infty : \mathfrak{m}(V) \to \mathfrak{m}(W) : [v_{i,j}] \to [\varphi(v_{i,j})].$$

Given a matrix

$$\varphi = [\varphi_{i,j}] \in M_n(L(V, W))$$

we define a linear mapping $\varphi : V \to M_n(W)$ by letting $\varphi(v) = [\varphi_{i,j}(v)]$. We use this to make the identification

$$M_n(L(V, W)) \cong L(V, M_n(W)), \tag{5}$$

and similarly we have the identification

$$\mathfrak{m}(L(V, W)) \cong L(V, \mathfrak{m}(W)). \tag{6}$$

Given a non-degenerate pairing of vector spaces

$$\langle \cdot, \cdot \rangle : V \times W \to \mathbb{C}, \tag{7}$$

each element $v \in V$ (respectively, $w \in W$) determines a linear functional $v : W \to \mathbb{C}$ (respectively, $w : V \to \mathbb{C}$) by

$$v(w) = w(v) = \langle v, w \rangle.$$

The pairing (7) determines the *scalar pairings*

$$\langle \cdot, \cdot \rangle : M_n(V) \times M_n(W) \to \mathbb{C} \tag{8}$$

and

$$\langle \cdot, \cdot \rangle : \mathfrak{m}(V) \times \mathfrak{m}(W) \to \mathbb{C} \tag{9}$$

where

$$\langle v, w \rangle = \sum_{i,j} \langle v_{i,j}, w_{i,j} \rangle$$

On the other hand it also determines the *matrix pairings*

$$\langle\langle \cdot, \cdot \rangle\rangle : M_m(V) \times M_n(W) \to M_{m \times n} \tag{10}$$

and

$$\langle\langle \cdot, \cdot \rangle\rangle : \mathfrak{m}(V) \times \mathfrak{m}(W) \to \mathfrak{m}_{\infty \times \infty} \tag{11}$$

where

$$\langle\langle v, w \rangle\rangle = [\langle v_{i,j}, w_{k,l} \rangle] = w_m(v) = v_n(w) \tag{12}$$

for $m, n = 1, \ldots, \infty$. In the latter formulae we regard $v$ and $w$ as linear mappings $v{:}W \to M_m$ and $w{:}V \to M_n$, or mappings into $\mathfrak{m}$, respectively.

## 3. Matrix gauges and matrix convexity

A *gauge* on a vector space $V$ is a function

$$\gamma : V \to [0, \infty]$$

such that

- **G1.** $\gamma(v + w) \leq \gamma(v) + \gamma(w)$, and
- **G2.** $\gamma(\alpha v) \leq |\alpha| \gamma(v)$

for all $v, w \in V$ and $\alpha \in \mathbb{C}$.We say that a gauge $p$ is *faithful* if $p(v) = 0$ implies $v = 0$. A gauge $\rho$ is a *seminorm* on $V$ if $\rho(v) < \infty$ for all $v \in V$. A faithful gauge $p$ determines a norm on the space

$${}_pV = \{v \in V : p(v) < \infty\}, \tag{13}$$

whereas a seminorm $\rho$ determines a norm $\|\cdot\|_\rho$ on the quotient vector space

$$V_\rho = V/N_\rho, \tag{14}$$

where

$$N_\rho = \{v \in V : \rho(v) = 0\}.$$

Letting $\pi_\rho$ be the quotient mapping of $V$ onto $V_\rho$, we have that

$$\rho(v) = \|\pi_\rho(v)\|_\rho.$$

If $\gamma$ is a gauge on $V$, then the *unit set*

$$B^\gamma = \{v : \gamma(v) \leq 1\}$$

is *absolutely convex,* i.e., if we are given $v_i \in B^\gamma$, and $\alpha_i \in \mathbb{C}$ $(1 \leq i \leq n)$ for which $\sum |\alpha_i| \leq 1$, then $\sum \alpha_i v_i \in B^\gamma$. Conversely, given an absolutely convex set $B \subseteq V$, the corresponding Minkowski functional $\gamma^B$ is a gauge on $V$. We recall that

$$\gamma^B(v) = \inf\{\lambda > 0 : v \in \lambda B\},$$

where we let $\gamma^B(v) = \infty$ if $v \notin \lambda B$ for any $\lambda > 0$.

Given an arbitrary vector space $V$, a *matrix* (or *operator*) *gauge* $\gamma_{*}$ *on* $V$ is a sequence of gauges

$$\gamma_n : M_n(V) \to [0, \infty]$$

such that for any $v \in M_m(V)$, $w \in M_n(V)$, $\alpha \in M_{n,m}$, and $\beta \in M_{m,n}$,

- **MG1.** $\gamma_{m+n}(v \oplus w) = \max\{\gamma_m(v), \gamma_n(w)\}$, and
- **MG2.** $\gamma_n(\alpha v \beta) \leq \|\alpha\| \gamma_m(v) \|\beta\|$.

Owing to **MG1**, we have a well-defined gauge $\gamma_\infty$ on $\mathfrak{m}(V)$ determined by the relation

$$\gamma_\infty(v) = \gamma_n(v),$$

when $v \in M_n(V)$. We say that elements $v_i \in \mathfrak{m}(V)$ $(i = 1 \ldots, n)$ are *orthogonal* if there exist orthogonal projections $e_i \in \mathfrak{m}$ for which $e_i v_i e_i = v_i$.

**Lemma 3.1.** *If $\gamma_{*}$ is a matrix gauge on $V$, then the corresponding gauge $\gamma_\infty$ satisfies the following properties:*

**$\mathfrak{m}$G1.** *If $v$ and $w$ are orthogonal in $\mathfrak{m}(V)$, then $\gamma_\infty(v + w) = \max\{\gamma_\infty(v), \gamma_\infty(w)\}$, and*

**$\mathfrak{m}$G2.** $\gamma_\infty(\alpha v \beta) \leq \|\alpha\| \gamma_\infty(v) \|\beta\|$ *for all $\alpha, \beta \in \mathfrak{m}, v \in M_m(V)$, and $m \in \mathbb{N}$.*

*Proof.* Given $v, w \in \mathfrak{m}(V)$ and orthogonal projections $e, f \in \mathfrak{m}$ with $eve = v$ and $fwf = w$, we fix an integer $m$ with $v, w \in M_m(V)$ and $e, f \in M_m$. We may choose partial isometries $\alpha$ and $\beta \in M_m$ which are equivalences of $e$ and $f$ with the projections $\varepsilon^{(k)}$ and $\varepsilon^{(l)}$, where $k$ and $l$ are the dimensions of these projections. It follows that $\alpha v \alpha^* \in M_k(V)$, $\beta w \beta^* \in M_l(V)$, and

$$v + w = \begin{bmatrix} \alpha^* & \beta^* \end{bmatrix} (\alpha v \alpha^* \oplus \beta w \beta^*) \begin{bmatrix} \alpha \\ \beta \end{bmatrix}.$$

Noting that

$$\left\| \begin{bmatrix} \alpha \\ \beta \end{bmatrix} \right\|^2 = \left\| \begin{bmatrix} \alpha^* & \beta^* \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \end{bmatrix} \right\| = \|e + f\| \leq 1,$$

it follows that

$$\gamma_\infty(v + w) \leq \max\{\gamma_\infty(v), \gamma_\infty(w)\}.$$

The remaining verifications are trivial. □

We say that a gauge $\gamma$ on $\mathfrak{m}(V)$ that satisfies $\mathfrak{m}$**G1** and $\mathfrak{m}$**G2** is an $\mathfrak{m}$-*module gauge.* It is evident that given such a gauge, the restrictions $\gamma_n = \gamma|_{M_n(V)}$ determine a matrix gauge $\gamma_{*}$ on $V$, and that we have a one-to-one relation between matrix gauges $\gamma_{*}$ on $V$ and $\mathfrak{m}$-module gauges $\gamma$ on $\mathfrak{m}(V)$.

We say that a graded set $B_{*}$ in $M_{*}(V)$, is *absolutely matrix convex* if for all $m, n \in \mathbb{N}$,

- **MC1.** $B_m \oplus B_n \subseteq B_{m+n}$, and
- **MC2.** $\alpha B_m \beta \subseteq B_n$ for any $n$ by $m$ and $m$ by $n$ contractions $\alpha$ and $\beta$.

Given a matrix gauge $\gamma_{*}$ on $V$, the sequence of unit sets $B^{\gamma}_{*} = (B^{\gamma_n})$ is an absolutely matrix convex set. Conversely, given an absolutely matrix convex graded set $B_{*} = (B_n)$, the corresponding Minkowski norms $\gamma^{B_n}$ associated with the convex sets $B_n \subseteq M_n(V)$ determine an operator seminorm $\gamma^B_{*}$. It is a simple matter to verify that $B_\infty = \cup B_n$ is the unit ball of the $\mathfrak{m}$-module gauge $\gamma_\infty$.

We say that a set $B \subseteq \mathfrak{m}(V)$ is *absolutely* $\mathfrak{m}$-*convex*if

- $\mathfrak{m}$**C1.** for any orthogonal elements $v, w \in B$, we have that $v + w \in B$, and
- $\mathfrak{m}$**C2.** $\alpha B\beta \subseteq B$ for any contractions $\alpha, \beta \in \mathfrak{m}$.

The argument used in Lemma 3.1 shows that the mapping $B_{\circledast} \mapsto B_{\infty} = \cup B_n$ provides a one-to-one correspondence between matrix convex graded sets for $V$ and $\mathfrak{m}$-convex sets in $\mathfrak{m}(V)$.

It is evident that an intersection of absolutely $\mathfrak{m}$-convex sets is an $\mathfrak{m}$-convex set, and thus if we are given a set $S \subseteq \mathfrak{m}(V)$, we may define the *$\mathfrak{m}$-convex hull* $h_{op}(S)$ of $S$ to be the smallest absolutely $\mathfrak{m}$-convex set containing $S$. We are indebted to B. E. Johnson for the following result.

**Lemma 3.2.** *Given an arbitrary subset* $S \subseteq \mathfrak{m}(V)$,

$$h_{op}(S) = \left\{ \sum \alpha_i v_i \beta_i : v_i \in S \right\}$$

*where* $\alpha_1, \ldots, \alpha_n$, *and* $\beta_1, \ldots, \beta_n \in \mathfrak{m}$ *satisfy*

$$\sum \alpha_i \alpha_i^*, \sum \beta_i^* \beta_i \leq 1.$$

*Proof.* We have that

$$\sum \alpha_i v_i \beta_i = [\alpha_1 \ldots \alpha_n] (v_1 \oplus \ldots \oplus v_n) \begin{bmatrix} \beta_1 \\ \vdots \\ \beta_n \end{bmatrix},$$

and thus it is evident that if these elements satisfy the indicated relations, we have that $\sum \alpha_i v_i \beta_i \in h_{op}(S)$. Conversely, given contractions $\gamma$ and $\delta$, we have that

$$\gamma (\sum \alpha_i v_i \beta_i) \delta = [\gamma \alpha_1 \ldots \gamma \alpha_n] (v_1 \oplus \ldots \oplus v_n) \begin{bmatrix} \beta_1 \delta \\ \vdots \\ \beta_n \delta \end{bmatrix}$$

where

$$\sum \gamma \alpha_i \alpha_i^* \gamma^* = \gamma^* \left( \sum \alpha_i \alpha_i^* \right) \gamma$$

is again a contraction, and the corresponding relation holds for the $\beta_i \delta$ terms. If we are given orthogonal projections $e, f$ and we have that $v = \sum \alpha_i v_i \beta_i$ and $w = \sum \gamma_j w_j \delta_j$ satisfy $eue = u$ and $fvf = v$, then we have that

$$\sum e \alpha_i \alpha_i^* e + \sum f \gamma_j \gamma_j^* f \leq 1$$

and

$$\sum e \beta_i^* \beta_i e + \sum f \delta_j^* \delta_j f \leq 1,$$

and thus

$$u + v = \sum e \alpha_i v_i \gamma_i e + \sum f \beta_j w_j \delta_j f$$

is a decomposition of the same type. □

Let us suppose that $\gamma$ is a matrix gauge on $V$. Given a finite set $\mathfrak{s}_0$ with $n$ elements, we may use a bijection of $\mathfrak{s}_0$ onto $\{1, \ldots, n\}$ to identify $M_{\mathfrak{s}_0}(V)$ with $M_n(V)$, and thus provide $M_{\mathfrak{s}_0}(V)$ with a gauge $\gamma_{\mathfrak{s}_0}$. This does not depend on the particular bijection since a permutation of $\{1, \ldots, n\}$ determines a unitary $\alpha : \mathbb{C}^n \to \mathbb{C}^n$, and from $\mathfrak{m}$**G2** the mapping

$$v \mapsto \alpha^* v \alpha$$

leaves the gauge $\gamma$ on $M_n(V)$ invariant. If $\mathfrak{s}$ is an arbitrary set, we may define a gauge on $\mathfrak{m}_{\mathfrak{s}}(V)$ by letting

$$\gamma_{\mathfrak{s}}(v) = \sup\{\gamma_{\mathfrak{s}_0}(v_{\mathfrak{s}_0})\},$$

where the supremum is taken over all finite subsets $\mathfrak{s}_0$ of $\mathfrak{s}$. In particular we may use the norms on the spaces $M_{\mathfrak{s}_0}$ to define a norm on $\mathfrak{m}_{\mathfrak{s}}$.

Given a matrix gauge $\gamma$ on a vector space $V$ and a matrix $v = [v_{i,j}] \in M_n(V)$, we have from (3) and (4) and **MC2** the important constraints

$$\begin{aligned} \gamma_1(v_{i,j}) &= \gamma_1(E_i^{(n)} v E_j^{(n)*}) \\ &\leq \gamma_n(v) \\ &= \gamma_n(\sum E_i^{(n)*} v_{i,j} E_j^{(n)}) \\ &\leq n^2 \max\{\gamma_1(v_{i,j})\}. \end{aligned} \tag{15}$$

It follows that if $p_*$ is a matrix gauge for which $p_1$ is faithful, then the same is true for each of the $p_n$ and thus for $p_\infty$, and similarly if $\rho_*$ is a matrix gauge for which $\rho_1$ is a seminorm, i.e., it is finite, then that is also the case for each $\rho_n$ ($n \in \mathbb{N} \cup \{\infty\}$). In the latter case we say that $\rho$ is a *matrix* or *operator seminorm*. If $\rho$ is a matrix seminorm and $N_\rho = \{v \in V : \rho_1(v) = 0\}$, we have that $\rho$ determines an operator norm $\|\cdot\|_*^\rho$ on $V_\rho = V/N_\rho$, and thus $V_\rho$ is an operator space with

$$\|(\pi_\rho)_n(v)\|_n^\rho = \rho_n(v).$$

## 4. The bipolar theorem.

If $V$ is a locally convex topological vector space, then the *canonical* topology on $M_n(V)$ is that determined by the linear isomorphism (1), i.e., we have that a net $v_\gamma \in M_n(V)$ converges to an element $v \in M_n(V)$ if and only if $(v_\gamma)_{i,j} \to v_{i,j}$.

If $V$ is a locally convex vector space, a gauge $\gamma$ on $V$ is *lower semicontinuous* if the corresponding unit set $B^\gamma$ is closed. The mappings $\gamma \mapsto B^\gamma$ and $B \mapsto \gamma^B$ determine one-to-one correspondences between the lower semi-continuous gauges $\gamma$ and the weakly closed absolutely convex subsets $B \subseteq V$. On the other hand, we have that $B^\gamma$ is a neighborhood of 0 if and only if $\gamma$ is a continuous seminorm (see [23], p. 14).

Two locally convex spaces $V$ and $W$ are *in duality* (or a *dual pair*) if one is given a distinguished pairing

$$\langle \cdot, \cdot \rangle : V \times W \to \mathbb{C}$$

such that all the continuous functionals on $V$ are given by elements of $W$, and vice versa. If this is the case, we have that $M_n(V)$ and $M_n(W)$ are in duality under the scalar pairing (8). Both the scalar pairing and the matrix pairing (10) determine the same topologies on $M_n(V)$ and $M_n(W)$, which we refer to as the *weak* topologies. We define the *weak* topology on $\mathfrak{m}$ to be that determined by the scalar pairing

$$\mathfrak{m} \times \mathfrak{m} \to \mathbb{C} : (\alpha, \beta) \mapsto \langle \alpha, \beta \rangle. \tag{16}$$

We may similarly use the scalar pairing (9), or letting $\mathfrak{m}$ have the weak topology, the matrix pairing (11) to define the *weak* topologies on $\mathfrak{m}(V)$ and $\mathfrak{m}(W)$. Using (12), it is easy to see that the bilinear mapping

$$\langle\langle \cdot, \cdot \rangle\rangle : V \times M_n(W) \to M_n$$

determines all the weakly continuous mappings $\varphi : V \to M_n$, i.e., we have

$$M_n(W) \cong \mathcal{C}(V, M_n). \tag{17}$$

Given a subset $D \subseteq V$, the *absolute polar* of $D$ is defined by

$$D^\circ = \{w \in W : |\langle v, w\rangle| \leq 1 \text{ for all } v \in D\}.$$

The classical bipolar theorem implies that $D^{\circ\circ}$ is the smallest weakly closed absolutely convex set containing $D$, and thus $D \mapsto D^\circ$ is a one-to-one correspondence between the weakly closed absolutely convex sets of $V$ and $W$.

Given a graded set $D_{\circledast} \subseteq M_{\circledast}(V)$, the (*absolute*) *operator polar* $D_{\circledast}^{\circledcirc} \subseteq M_{\circledast}(W)$, is defined by

$$D_n^{\circledcirc} = \{w \in M_n(W) : \|\langle\langle v, w\rangle\rangle\| \leq 1 \text{ for all } v \in D_r, \text{ and } r \in \mathbb{N}\}.$$

We note that

$$D_1^{\circledcirc} = D_1^\circ \tag{18}$$

since if $f \in V'$ satisfies $|\langle f, x\rangle| \leq 1$ for all $x \in D_1$, then $\|\langle\langle f, x\rangle\rangle\| \leq 1$ for all $x \in D_n$. To see this let us suppose that $\xi, \eta \in \mathbb{C}^n$ are unit vectors. Then regarding these as unit column matrices, we have that

$$\langle\, \langle\langle f, x\rangle\rangle\, \eta|\xi\rangle = \langle f|\xi^* x\eta\rangle < 1$$

(see Lemma 5.2 for a more general result).

The following bipolar theorem is variation of [11]. Owing to the fact that we must allow infinite gauges in our applications, it is necessary to modify that argument, using a technique introduced in [13]. Unfortunately, we are not aware of a simple argument that would reduce this result to the bipolar theorem for non-absolutely matrix convex sets proved in [13]. The reader may prefer to initially skip over the rather lengthy proof.

We say that $D_{\circledast}$ is *weakly closed* if that is the case for each set $D_n$.

**Proposition 4.1.** *Suppose that $V$ and $W$ are dual locally convex spaces. Given any graded set $D_{\circledast} \subseteq M_{\circledast}(V)$, we have that $D_{\circledast}^{\circledcirc\circledcirc}$ is the smallest weakly closed absolutely matrix convex set containing $D_{\circledast}$.*

*Proof.* It suffices to prove that if $D_{\circledast}$ is absolutely matrix convex and weakly closed, then $D_{\circledast}^{\circledcirc\circledcirc} = D_{\circledast}$. Thus we must prove that if $v_0 \notin D_n$, then $v_0 \notin D_n^{\circledcirc\circledcirc}$. To do this we will show that there exists a $w_0 \in M_n(W)$ for which $w_0 \in (D_n^{\circledcirc})$, i.e.,

$$\|\langle\langle v, w_0\rangle\rangle\| \leq 1,$$

for all $v \in D_r$ and $r \in \mathbb{N}$, but

$$\|\langle\langle v_0, w_0\rangle\rangle\| > 1$$

Letting $\gamma$ be the matrix gauge of $D_{\circledast}$ and using the identification (17), it suffices to find a mapping $\varphi : V \to M_n$ for which

$$\|\varphi_r(v)\| \leq 1 \tag{19}$$

for all $v \in D_r$ and $r \in \mathbb{N}$, but

$$\|\varphi_n(v_0)\| > 1. \tag{20}$$

Using the classical bipolar theorem, we have that $v_0 \notin D_n^{\circ\circ}$, i.e., we may find a functional $F \in M_n(V)'$ for which

$$|F|_{D_n}| \leq 1 < |F(v_0)|\,. \tag{21}$$

As in [13] we will extract the desired matrix valued function $\varphi$ from a perturbation of $F$.

Our argument depends on a simple convexity result. Let us suppose that $\mathcal{E}$ is a cone of real continuous affine functions on a compact convex subset $K$ of a topological vector space $E$, and that for each $e \in \mathcal{E}$, there is a corresponding point $k_e \in K$ with $e(k_e) \geq 0$. Then there is a point $k_0 \in K$ for which $e(k_0) \geq 0$ for all $e \in \mathcal{E}$. A simple proof of this result may be found in [13], Lemma 5.2.

Following [11], we claim that there exist states $p_0$ and $q_0$ on $M_n$ such that

$$|F(\alpha v \beta)| \leq p_0(\alpha\alpha^*)^{1/2}\gamma_r(v)q_0(\beta^*\beta)^{1/2} \tag{22}$$

for all $\alpha \in M_{n,r}, \beta \in M_{r,n}$, and $v \in M_r(V)$ for which $\gamma_r(v) < \infty$, with $r \in \mathbb{N}$ arbitrary. If $\gamma_r(v) = 0$, then

$$\gamma_n(\alpha v \beta) \leq \|\alpha\| \, \gamma_r(v) \, \|\beta\| = 0,$$

i.e., $\lambda(\alpha v \beta) \in D_n$ for all scalars $\lambda > 0$. Since we chose $F$ with $|F|_{D_n}| \leq 1$, we have $\lambda |F(\alpha v \beta)| \leq 1$ for all scalars $\lambda > 0$, hence $F(\alpha v \beta) = 0$ and (22) is trivial. Dividing by a constant, we may assume that $\gamma_r(v) = 1$. Summarizing, our task is to find states $p_0$ and $q_0$ on $M_n$ such that for all $v \in M_r(V)$ with $\gamma_r(v) = 1$, we havethat

$$|F(\alpha v \beta)| \leq p_0(\alpha\alpha^*)^{1/2}q_0(\beta^*\beta)^{1/2}. \tag{23}$$

It suffices to find states $p_0, q_0$ such that

$$\operatorname{Re} F(\alpha v \beta) \leq p_0(\alpha\alpha^*)^{1/2}q_0(\beta^*\beta)^{1/2}$$

since we can then replace $\alpha$ by $e^{i\theta}\alpha$ for a suitable $\theta \in [0, 2\pi]$. In turn, it is enough to prove that

$$\operatorname{Re} F(\alpha v \beta) \leq (1/2)[p_0(\alpha\alpha^*) + q_0(\beta^*\beta)]. \tag{24}$$

To see this, we replace $\alpha$ by $t^{1/2}\alpha$ and $\beta$ by $t^{-1/2}\beta$ for $t > 0$. It follows that

$$\operatorname{Re} F(\alpha v \beta) \leq (1/2)[tp_0(\alpha\alpha^*) \; + t^{-1}q_0(\beta^*\beta)],$$

and minimizing for $t > 0$ (or simply letting $t = p_0(\alpha\alpha^*)^{-1/2}q_0(\beta^*\beta)^{1/2}$ for $\alpha \neq 0$), we obtain (23).

Letting $S_n$ be the set of states on $M_n$, $S = S_n \times S_n$ is a compact and convex subset of $(M_n \oplus M_n)^*$. We write $A(S)$ for the continuous affine functions on $S$. Given $\alpha \in M_{n,r}$, $\beta \in M_{r,n}$ and $v \in M_r(V)$ with $\gamma_r(v) = 1$, we may define a corresponding function $e_{\alpha,v,\beta} \in A(K)$ by

$$e_{\alpha,v,\beta}(p, q) = p(\alpha\alpha^*) + q(\beta^*\beta) - 2ReF(\alpha v \beta).$$

Letting $\mathcal{E}$ denote the collection of all such functions, we must show that there is a point $(p_0, q_0) \in S$ for which $e(p_0, q_0) \geq 0$ for all $e \in \mathcal{E}$. But we have that

a) Each function $e \in \mathcal{E}$ is non-negative at some point $(p_e, q_e) \in S$. To see this suppose that $e = e_{\alpha,v,\beta}$, and select states $p_e$ and $q_e$ with $p_e(\alpha\alpha^*) = \|\alpha\alpha^*\| = \|\alpha\|^2$, and $q_e(\beta^*\beta) = \|\beta\|^2$. Then we have

$$e_{\alpha,v,\beta}(p_e, q_e) = \|\alpha\|^2 + \|\beta\|^2 - 2ReF(\alpha v \beta) \geq 0$$

since

$$\operatorname{Re} F(\alpha v \beta) \leq |F(\alpha v \beta)| \leq \|\alpha v \beta\| \leq \|\alpha\|\|\beta\| \leq (1/2)[\|\alpha\|^2 + \|\beta\|^2].$$

b) The collection $\mathcal{E}$ is a cone of functions, i.e., $\mathcal{E}+\mathcal{E}\subseteq\mathcal{E}$, and $\lambda\mathcal{E}\subseteq\mathcal{E}$ for $\lambda\geq 0$. The second assertion is trivial. For the first we note that

$$e_{\alpha,v,\beta}+e_{\alpha',v',\beta'}=e_{\alpha'',v'',\beta''}$$

where $\alpha''=\left[\begin{array}{cc}\alpha & \alpha'\end{array}\right]$, $\beta''=\left[\begin{array}{c}\beta \\ \beta'\end{array}\right]$, and $v''=v\oplus v'\in M_{r+r'}$ satisfies $\gamma_{r+r'}(v\oplus v')=1$.

¿From the convexity result, there exists a point $(p_0,q_0)$ at which all of the functions in $\mathcal{E}$ are positive. Thus we have proved (22). We claim that we may perturb these states so that they are faithful. We recall that the normalized trace $\tau$ on $M_n$ is faithful, i.e., we have that $\tau(\alpha^*\alpha)=0$ implies that $\alpha=0$. Given $0<\varepsilon<1$, it follows that $p=(1-\varepsilon)p_0+\varepsilon\tau$ and $q=(1-\varepsilon)q_0+\varepsilon\tau$ are faithful. Letting $G=(1-\varepsilon)F$, we have from (22) that if $\gamma_r(v)<\infty$,

$$\begin{aligned}|G(\alpha v\beta)| &\leq (1-\varepsilon)p_0(\alpha\alpha^*)^{1/2}q_0(\beta^*\beta)^{1/2}\gamma_r(v)\\ &\leq \frac{(1-\varepsilon)}{2}[p_0(\alpha\alpha^*)+q_0(\beta^*\beta)]\gamma_r(v)\\ &\leq \frac{1}{2}[p(\alpha\alpha^*)+q(\beta^*\beta)]\gamma_r(v).\end{aligned}$$

Replacing $\alpha$ by $t^{1/2}\alpha$ and $\beta$ by $t^{-1/2}\alpha$, where $t$ is a positive scalar, and then minimizing, we conclude that

$$|G(\alpha v\beta)|\leq p(\alpha\alpha^*)^{1/2}\gamma_r(v)q(\beta^*\beta)^{1/2}. \tag{25}$$

On the other hand, if we let $\varepsilon$ be sufficiently small, we have from (21) that

$$|G|_{D_n}|\leq 1<|G(v_0)|.$$

Applying the GNS theorem, we have corresponding faithful representations $\pi$ and $\theta$ of $M_n$ on finite dimensional Hilbert spaces $H$ and $K$, respectively, with separating and cyclic vectors $\xi_0\in H$ and $\eta_0\in K$ satisfying $p(\alpha)=\langle\pi(\alpha)\xi_0|\xi_0\rangle$ and $q(\alpha)=\langle\theta(\alpha)\eta_0|\eta_0\rangle$, respectively, for all $\alpha\in M_n$.

Given a row matrix $\alpha=[\alpha_1,\alpha_2,\ldots,\alpha_n]\in M_{1,n}$, we define $\tilde{\alpha}\in M_n$ by

$$\tilde{\alpha}=\left[\begin{array}{ccc}\alpha_1 & \alpha_2 & \ldots\\ 0 & 0 & \ldots\\ \ldots & \ldots & \ldots\end{array}\right].$$

We let $\tilde{M}_{1,n}$ be the linear space of all such $n$ by $n$ matrices, and we let $H_0=\pi(\tilde{M}_{1,n})\xi_0\subseteq H$ and $K_0=\theta(\tilde{M}_{1,n})\eta_0\subseteq K$. Owing to the fact that $\xi_0$ and $\eta_0$ are separating, $H_0$ and $K_0$ are $n$-dimensional spaces.

Fixing an element $v\in V$, the sesquilinear form $B_v$ defined on $K_0\times H_0$ by

$$B_v(\theta(\tilde{\beta})\eta_0,\pi(\tilde{\alpha})\xi_0)=G(\alpha^*v\beta)$$

is well-defined since if, for example, $\theta(\tilde{\beta})\eta_0=0$, then since $\eta_0$ is separating and $\theta$ is a faithful representation, we have that $\tilde{\beta}=0$ and $G(\alpha^*v\beta)=0$. Thus there exists a unique linear map $\varphi(v):K_0\to H_0$ for which

$$G(\alpha^*v\beta)=\langle\varphi(v)\theta(\tilde{\beta})\eta_0\mid\pi(\tilde{\alpha})\xi_0\rangle.$$

It is a simple matter to verify that the corresponding map $\varphi:V\to\mathcal{B}(K_0,H_0)$ is linear. Since $H_0$ and $K_0$ are $n$-dimensional, we may identify each of these spaces with $\mathbb{C}^n$, and $\varphi$ with a mapping $\varphi:V\to M_n$.

Given a matrix $v \in M_n(V)$, we have from (4)

$$\begin{aligned} G(v) &= \Sigma\langle \varphi(v_{ij})\theta(\tilde{E}_j^{(n)})\eta_0 \mid \pi(\tilde{E}_i^{(n)})\xi_0\rangle \\ &= \langle \varphi_n(v)\eta \mid \xi\rangle, \end{aligned} \tag{26}$$

where

$$\eta = \begin{pmatrix} \theta(\tilde{E}_1^{(n)})\eta_0 \\ \vdots \\ \theta(\tilde{E}_n^{(n)})\eta_0 \end{pmatrix}, \ \xi = \begin{pmatrix} \pi(\tilde{E}_1^{(n)})\xi_0 \\ \vdots \\ \pi(\tilde{E}_n^{(n)})\xi_0 \end{pmatrix} \in \mathbb{C}^n$$

satisfy

$$||\xi||^2 = \Sigma||\pi(\tilde{E}_j^{(n)})\xi_0||^2 = \Sigma p(E_j^{(n)*}E_j^{(n)}) = p(\varepsilon^{(n)}) = 1,$$

and similarly, $||\eta||^2 = 1$.

To show that $\varphi$ satisfies (19), we must prove that if $v \in M_r(V)$ and $\gamma_r(v) < \infty$, then

$$|\langle \varphi_r(v)\eta_1 \mid \xi_1\rangle| \leq \gamma_r(v)||\eta_1||||\xi_1||$$

for unit vectors $\xi_1$ and $\eta_1$ in $(\mathbb{C}^n)^r$. Letting

$$\xi_1 = \begin{pmatrix} \pi(\tilde{\alpha}_1)\xi_0 \\ \vdots \\ \pi(\tilde{\alpha}_r)\xi_0 \end{pmatrix}, \ \eta_1 = \begin{pmatrix} \theta(\tilde{\beta}_1)\eta_0 \\ \vdots \\ \theta(\tilde{\beta}_r)\eta_0 \end{pmatrix}$$

where $\alpha_i$, $\beta_j \in M_{1,n}$, we have that

$$||\xi_1||^2 = \Sigma||\pi(\tilde{\alpha}_i)\xi_0||^2 = \Sigma p(\alpha_i^*\alpha_i) = p(\alpha^*\alpha),$$

and similarly $||\eta_1||^2 = q(\beta^*\beta)$ , where

$$\alpha = \begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_r \end{bmatrix}, \ \beta = \begin{bmatrix} \beta_1 \\ \vdots \\ \beta_r \end{bmatrix} \in M_{r,n}.$$

It follows that

$$\begin{aligned} \langle \varphi_r(v)\eta_1 \mid \xi_1\rangle &= \Sigma\langle \varphi_0(v_{i,j})\theta(\tilde{\beta}_j)\eta_0 \mid \pi(\tilde{\alpha}_i)\xi_0\rangle \\ &= \Sigma G(\alpha_i^* v_{ij}\beta_j) \\ &= G(\alpha^* v\beta), \end{aligned}$$

and thus if $v \in D_r$,

$$\begin{aligned} |\langle \varphi_r(v)\eta_1 \mid \xi_1\rangle| &\leq |G(\alpha^* v\beta)| \\ &\leq p(\alpha\alpha^*)^{1/2}\gamma_r(v)q(\beta^*\beta)^{1/2} \\ &= \gamma_r(v)||\xi_1||||\eta_1|| \\ &\leq 1. \end{aligned}$$

On the other hand we have that $v_0 \in M_n(V)$ satisfies

$$1 < G(v_0) = \langle \varphi_n(v_0)\eta \mid \xi\rangle,$$

and since that $\eta$ and $\xi$ are unit vectors, we have proved (20). □

Given a subset $S \subseteq \mathfrak{m}(V)$, we define its $\mathfrak{m}$-*polar* $S^{\circledcirc} \subseteq \mathfrak{m}(W)$ by

$$S^{\circledcirc} = \{w \in \mathfrak{m}(W) : \|\langle\langle v, w\rangle\rangle\| \leq 1 \text{ for all } v \in S\}.$$

We claim that $S^{\circledcirc}$ is $\mathfrak{m}$-convex. If $w, w' \in S^{\circledcirc}$ and $e, f \in \mathfrak{m}$ are orthogonal projections with $ewe = w$ and $fw'f = w'$, then given $v \in S_n = S \cap M_n$, we have that $\tilde{e} = \varepsilon^{(n)} \otimes e$ and $\tilde{f} = \varepsilon^{(n)} \otimes f$ are orthogonal projections in $\mathfrak{m} \otimes \mathfrak{m}$, satisfying $\tilde{e}\langle\langle v, w\rangle\rangle\tilde{e} = \langle\langle v, w\rangle\rangle$ and $\tilde{f}\langle\langle v, w'\rangle\rangle\tilde{f} = \langle\langle v, w'\rangle\rangle$. It follows that

$$\begin{aligned} \|\langle\langle v, ewe + fw'f\rangle\rangle\| &= \left\|\tilde{e}\langle\langle v, w\rangle\rangle\tilde{e} + \tilde{f}\langle\langle v, w'\rangle\rangle\tilde{f}\right\| \\ &= \max\{\|\langle\langle v, w\rangle\|, \|\langle\langle v, w'\rangle\rangle\|\} \leq 1. \end{aligned}$$

On the other hand if $\alpha, \beta \in \mathfrak{m}$ are contractions, then that is also the case for $\tilde{\alpha} = \varepsilon^{(n)} \otimes \alpha$ and $\beta^{(n)} \otimes \beta$, and thus

$$\|\langle\langle v, \alpha v \beta\rangle\rangle\| = \left\|\tilde{\alpha}\langle\langle v, w\rangle\rangle\tilde{\beta}\right\| \leq \|\langle\langle v, w\rangle\rangle\| \leq 1.$$

**Corollary 4.1.** *Given a subset $S \subseteq \mathfrak{m}(V)$, $S^{\circledcirc\circledcirc}$ is the smallest weakly closed $\mathfrak{m}$-convex set containing $S$.*

*Proof.* If we let $B$ be the smallest $\mathfrak{m}$-convex set containing $S$, the weak closure $\bar{B}$ is obviously $\mathfrak{m}$-convex, and thus our task is to show that $B^{\circledcirc\circledcirc} = \bar{B}$. The sets

$$B_n = \varepsilon^{(n)} B \varepsilon^{(n)} \subseteq M_n(V)$$

determine an absolutely matrix convex set $B_{\circledast}$. The weak closure $\bar{B}$ is equal to $\cup \bar{B}_n$ since if $v_\lambda$ is a net in $\cup B_n$ converging to $v \in \mathfrak{m}(V)$, then assuming that $v \in M_n(V)$,

$$v = \lim_\lambda \varepsilon^{(n)} v_\lambda \varepsilon^{(n)} \in \bar{B}_n.$$

Thus if $v_0 \notin \bar{B}$, then we may assume that for some $n \in \mathbb{N}$,

$$v_0 \in M_n(V) \backslash \bar{B}_n.$$

From the bipolar theorem there exists a $w_0 \in M_n(W)$ such that

$$\|\langle\langle v_0, w_0\rangle\rangle\| > 1 \geq \|\langle\langle v, w_0\rangle\rangle\|$$

for all $v \in B_r$, with $r \in \mathbb{N}$ arbitrary. Regarding $w_0$ as an element of $\mathfrak{m}(W)$, we have that $w_0 \in B_n^{\circledcirc}$, and thus $v_0 \notin B^{\circledcirc\circledcirc}$. □

Given a gauge $\gamma$ on $V$, we define the *dual gauge* $\gamma^\circ$ to be the gauge of the polar of the unit set of $\gamma$, or equivalently,

$$\gamma^\circ(f) = \sup\{|\langle v, f\rangle| : \gamma(v) \leq 1\}.$$

If $\gamma_{\circledast}$ is a matrix gaugeon $V$, the corresponding *dual matrix gauge* $\gamma_{\circledast}^{\circledcirc}$ is determined for $f \in M_n(V')$ by

$$\gamma_n^{\circledcirc}(f) = \sup\{\|\langle\langle v, f\rangle\rangle\| : \gamma_r(v) \leq 1, r \in \mathbb{N}\}$$

whereas, if $\gamma$ is an $\mathfrak{m}$-module gauge on $\mathfrak{m}(V)$, then the *dual* $\mathfrak{m}$-*module gauge* $\gamma^{\circledcirc}$ on $\mathfrak{m}(V')$ is defined by

$$\gamma^{\circledcirc}(f) = \sup\{\|\langle\langle v, f\rangle\rangle\| : \gamma(v) \leq 1, v \in \mathfrak{m}(V)\}.$$

Given a locally convex space $V$, we have that $V$ and $W = V' = \mathcal{C}(V, \mathbb{C})$ form a dual pair. An absolutely convex set $K \subseteq V$ determines a graded set $K_{\circledast}$, where

$$K_n = \begin{cases} K & \text{if } n = 1 \\ \{0\} & \text{if } n > 1 \end{cases}. \tag{27}$$

We define the *minimal envelope* $\hat{K} \subseteq M_{*}(V)$ to be $(K_{*})^{\circledcirc\circledcirc}$. It follows from the preceding theorem that $\hat{K}$ is the weak closure of the matrix convex set

$$\tilde{K} = \{\alpha(v_1 \oplus \ldots \oplus v_n)\beta : v_i \in K_{n_i}, \alpha, \beta \text{ contractions}\} \tag{28}$$

We define the *maximal envelope* $\check{K}$ to be the matrix polar of the classical polar $K^{\circ} \subseteq V'$. To be more precise, $\check{K} = (K^{\circ}_{*})^{\circledcirc}$, where

$$K^{\circ}_n = \begin{cases} K^{\circ} & \text{if } n = 1 \\ \{0\} & \text{if } n > 1 \end{cases}. \tag{29}$$

We claim that if $L_{*}$ is a weakly closed absolutely matrix convex set with $L_1 = K$, then

$$\hat{K} \subseteq L \subseteq \check{K}.$$

We have $L_{*} \supseteq K_{*}$ implies that $L^{\circledcirc} \subseteq K^{\circledcirc}$ and thus

$$L = L^{\circledcirc\circledcirc} \supseteq K^{\circledcirc\circledcirc} = \hat{K}.$$

On the other hand from (18), $L^{\circledcirc}_1 = L^{\circ}_1 = K^{\circ}_1$ implies that $L^{\circledcirc}_{*} \supseteq K^{\circ}_{*}$ and

$$L = L^{\circledcirc\circledcirc} \subseteq (K^{\circ}_{*})^{\circledcirc} = \check{K}.$$

If $\rho$ is the gauge of an absolutely convex weakly closed set $K$ in $V$, we let $\hat{\rho}$ and $\check{\rho}$ be the corresponding matrix gauges of $\hat{K}$ and $\check{K}$, respectively. It follows from above that if $\rho$ is any matrix gauge with $\rho_1 = \rho$, then

$$\check{\rho} \leq \rho \leq \hat{\rho}. \tag{30}$$

In particular, if $V$ is a normed space with the norm $\rho(v) = \|v\|$, the two *extremal* matrix norms on $V$ are defined for $v \in M_n(V)$ by

$$\|v\|_{\min} = \sup\{\|\langle\langle v, f\rangle\rangle\| : f \in V^*, \|f\| \leq 1\} = \check{\rho}_n(v) \tag{31}$$

(see [6], [2]) and

$$\|v\|_{\max} = \sup\{\|\langle\langle v, f\rangle\rangle\| : f \in \mathcal{B}(V, M_p), \|f\| \leq 1, p \in \mathbb{N}\} = \hat{\rho}_n(v). \tag{32}$$

(see [2]). The corresponding operator spaces are denoted by $\min V$ and $\max V$, respectively.

## 5. Local operator spaces

If $V$ is a locally convex space, then its topology is determined by the family $\mathfrak{N}(V)$ of continuous seminorms $\rho : V \to [0, \infty)$. In the remainder of the paper, we regard a locally convex space $V$ as in duality with its continuous dual $V'$, and we refer to the topology determined by $V'$ as the *weak* topology on $V$. Similarly, we define the *weak topology* on $\mathfrak{m}(V)$ to be that determined by either the scalar pairing $\mathfrak{m}(V) \times \mathfrak{m}(V') \to \mathbb{C}$, or equivalently, by the matrix pairing

$$\mathfrak{m}(V) \times \mathfrak{m}(V') \to \mathfrak{m} \otimes \mathfrak{m},$$

where we let $\mathfrak{m} \otimes \mathfrak{m} = \mathfrak{m}_{\infty \times \infty}$ have the weak topology determined by (16).

We recall that an *operator space* is a vector space $V$ together with a distinguished operator norm $\rho$. Given a locally convex space $V$, we define a *local operator structure*on $V$ to be a family $\mathfrak{R}$ of continuous operator seminorms $\rho$ on $V$, for which the seminorms $\rho_1$ generate the topology on $V$. We define the *matrix topology on* $\mathfrak{m}(V)$ *determined* (or *generated*) *by* $\mathfrak{R}$ to be the locally convex topology determined by the seminorms $\rho_\infty$ with $\rho \in \mathfrak{R}$. We define a *local operator space* to be a locally convex space $V$ together with a matrix topology on $\mathfrak{m}(V)$ which arises in the above manner. We shall say that a given local operator structure on a locally convex space is a *quantization* of that space. On the other hand when we shall on occassion refer to results in locally convex theory as being *classical.*

Let us suppose that $V$ is a local operator space generated by a family of continuous operator seminorms $\mathfrak{N}$. We say that an operator seminorm $\nu$ on $V$ is *matrix continuous* if the corresponding $\mathfrak{m}$-module seminorm

$$\nu_\infty : \mathfrak{m}(V) \to [0, \infty)$$

is continuous. Equivalently, we have that there exists an operator seminorm $\rho \in \mathfrak{N}$ and a constant $k > 0$ with $\nu_n \leq k\rho_n$ for all $n$. Letting $\mathfrak{N}_{op}(V)$ be the collection of all matrix continuous operator seminorms, it is evident that $\mathfrak{N}_{op}(V)$ and $\mathfrak{N}$ determine the same topology on $\mathfrak{m}(V)$. We say that a local operator space $V$ is *countably generated* if the matrix topology is generated by a countable family $\mathfrak{N}$ of operator seminorms. The collection $\mathfrak{N}_{op}(V)$ is partially ordered by the relation $\rho \leq \sigma$ if for each $v \in \mathfrak{m}(V)$ we have that $\rho_\infty(v) \leq \sigma_\infty(v)$.

Given two local operator spaces $V$ and $W$, we say that a linear mapping $\varphi : V \to W$ is *matrix* or *operator continuous* if the corresponding mapping $\varphi_\infty : \mathfrak{m}(V) \to \mathfrak{m}(W)$ is continuous in the matrix topologies. Thus we have that $\varphi$ is operator continuous if and only if for any operator seminorm $\sigma \in \mathfrak{N}_{op}(W)$, there is an operator seminorm $\rho \in \mathfrak{N}_{op}(V)$ and a constant $k > 0$ such that

$$\sigma_n(\varphi_n(v)) \leq k\rho_n(v).$$

for all $n \in \mathbb{N}$. Similarly we say that $\varphi$ is a *matrix* or *operator homeomorphism* if $\varphi_\infty$ is a homeomorphism in the matrix topologies. We let $\mathcal{C}_{op}(V, W)$ denote the vector space of all matrix continuous linear mappings $\varphi : V \to W$. If $V$ and $W$ are operator spaces, these notions coincide with those of *completely bounded* and *completely isomorphic* linear mappings $\varphi : V \to W$, respectively, i.e., in this case we have the linear space equality $\mathcal{C}_{op}(V, W) = \mathcal{B}_{op}(V, W)$.

It is apparent from (15) that the seminorms $\rho_n$ with $\rho \in \mathfrak{N}_{op}(V)$ determine the canonical topology on $M_n(V)$. Thus since the seminorm $\rho_\infty$ restricts to $\rho_n$ on $M_n(V)$, we conclude that the injection $M_n(V) \hookrightarrow \mathfrak{m}(V)$ is a homeomorphism onto its image. On the other hand, it is already apparent from the theory of operator spaces that the norm topology on $V$ does not determine that on $\mathfrak{m}(V)$. This is evident if one considers the row and column Hilbert spaces $H_c$ and $H_r$. Any completely bounded linear mapping $\varphi : H_c \to H_r$ is compact [8], Cor.4.5, hence if $H$ is infinite dimensional, $H_c$ and $H_r$ are not completely isomorphic.

We next show that any matrix topology on $\mathfrak{m}(V)$ contains the weak topology determined by the pairings (9) and (11), and thus $\mathfrak{m}$-polars of sets in $\mathfrak{m}(V')$ are automatically closed in $\mathfrak{m}(V)$. For this purpose we use several lemmas due to Roger Smith [27].

**Lemma 5.1.** *Given integers $m, n \in \mathbb{N}$ with $m \geq n$ and a vector $\eta \in \mathbb{C}^m \otimes \mathbb{C}^n$, there exists an isometry $\beta : \mathbb{C}^n \hookrightarrow \mathbb{C}^m$ and a vector $\bar{\eta} \in \mathbb{C}^n \otimes \mathbb{C}^n$ for which $(\beta \otimes I_n)(\bar{\eta}) = \eta$.*

*Proof.* There exist unique vectors $\eta_j \in \mathbb{C}^m$, $(j = 1, \ldots, n)$ with

$$\eta = \sum_{j=1}^n \eta_j \otimes \varepsilon_j^{(n)}.$$

Letting $F \subseteq \mathbb{C}^m$ be the subspace spanned by the vectors $\eta_j$, we have that $\dim F \leq n \leq m$. Thus we may find an isometry $\beta : \mathbb{C}^n \to \mathbb{C}^m$ with image containing $F$. For each $j$ we have a unique vector $\bar{\eta}_j \in \mathbb{C}^n$ for which $\beta(\bar{\eta}_j) = \eta_j$. Letting $\bar{\eta} = \sum_j \bar{\eta}_j \otimes \varepsilon_j^{(n)}$, we have that $\beta \otimes I_n(\bar{\eta}) = \eta$. □

**Lemma 5.2.** *Suppose that $\rho$ is a matrix gauge on a vector space $V$, and that $\varphi : V \to M_n$ is a linear map. If*

$$\|\varphi_n(v)\| \leq k\rho_n(v)$$

*for all $v \in M_n(V)$, we have that*

$$\|\varphi_r(v)\| \leq k\rho_r(v)$$

*for all $v \in M_r(V)$ with $r \in \mathbb{N}$ arbitrary.*

*Proof.* If $r \leq n$, then we have that

$$\|\varphi_r(v)\| = \|\varphi_n(v \oplus 0_{n-r})\| \leq k\rho_n(v \oplus 0_{n-r}) = k\rho_r(v).$$

Let us suppose that $r > n$. We may assume that $\rho_r(v) < \infty$. Given arbitrary unit vectors $\eta, \xi \in (\mathbb{C}^n)^m = \mathbb{C}^m \otimes \mathbb{C}^n$ we have from the above lemma that, there exist isometries $\alpha, \beta : \mathbb{C}^n \to \mathbb{C}^m$ and unit vectors $\bar{\xi}, \bar{\eta} \in \mathbb{C}^n \otimes \mathbb{C}^n$ for which $\xi = (\alpha \otimes I_n)(\bar{\xi})$ and $\eta = (\beta \otimes I_n)(\bar{\eta})$. We have that

$$\begin{aligned} |\langle \varphi_r(v)\eta|\xi\rangle| &= |\langle\, \varphi_r(v)(\beta \otimes I_n)(\bar{\eta})|(\alpha \otimes I_n)(\bar{\xi})\,\rangle| \\ &= |\langle\, \varphi_n(\alpha^* v \beta)\bar{\eta} \mid \bar{\xi}\,\rangle| \\ &\leq \|\, \varphi_n(\alpha^* v\beta)\| \\ &\leq k\rho_n(\alpha^* v \beta) \\ &\leq k\rho_r(v). \end{aligned}$$

Since $\eta$ and $\xi$ are arbitrary unit vectors, we conclude that

$$\|\varphi_r(v)\| \leq k\rho_r(v).$$

□

**Corollary 5.1.** *If $V$ is a local operator space, then the matrix topology contains the weak topology on $\mathfrak{m}(V)$.*

*Proof.* Given $f \in \mathfrak{m}(V')$, we may assume that $f \in M_n(V')$ for some $n \in \mathbb{N}$. By assumption the seminorms $\rho_1$ with $\rho \in \mathfrak{N}_{op}(V)$ determine the topology on $V$. It follows that there exists such a $\rho$ and a constant $k > 0$ with

$$\|f(v)\| \leq k\rho_1(v)$$

for all $v \in V$. It follows that if $v \in M_n(V)$ then

$$\begin{aligned} \|f_n(v)\| &\leq n^2 \max\{|f(v_{i,j})|\} \\ &\leq n^2 k\rho_1(v_{i,j}) \\ &\leq n^2 k\rho_n(v). \end{aligned}$$

Thus from Lemma 5.2, we have that

$$\|f_r(v)\| \leq n^2 k\rho_r(v)$$

for all $v \in M_r(V)$ with $r \in \mathbb{N}$ arbitrary. We conclude that $f$ is continuous as a linear functional on $\mathfrak{m}(V)$. □

We may associate *extremal quantizations* of a locally convex space $V$ in the following manner. Given a continuous seminorm $\rho \in \mathfrak{N}(V)$, we have that the unit ball $B^\rho$ is closed and convex, and thus it is closed in the weak topology determined by $V'$. Following the discussion of §4, we may associate with $\rho$ the minimal and maximal envelopes $\hat{\rho}$ and $\check{\rho}$, respectively, for each continuous seminorm $\rho \in \mathfrak{N}(V)$. We let $\max V$ and $\min V$ denote the local operator spaces determined by the $\hat{\rho}$ and $\check{\rho}$ for $\rho \in \mathfrak{N}(V)$ (the correspondence between sets and gauges is order inverting). If $V$ is a local operator space, then given $\rho \in \mathfrak{N}_{op}(V)$, we have from (30) a diagram of matrix continuous mappings

$$\max V \to V \to \min V.$$

Given a subspace $W$ of a locally convex operator space $V$, it is evident that the restrictions of operator seminorms in $\mathfrak{N}_{op}(V)$ determine a matrix topology on $\mathfrak{m}(W)$. If $W$ is closed in $V$, it follows from the inequalities (15) that $M_n(W)$ is closed in $M_n(V)$. We also have that if $V$ is complete, then it also follows from (15) that each of the spaces $M_n(V)$ is complete. It is often convenient to refer to a linear matrix homeomorphism of a local operator space $W$ into another local operator space $V$ as an "inclusion $W \subseteq V$ of local operator spaces".

If $\rho \in \mathfrak{N}_{op}(V)$, we let $\tilde{\rho}_n$ denote the corresponding quotient seminorm on

$$M_n(V/W) = M_n(V)/M_n(W). \tag{33}$$

These determine the *quotient* operator space structure on $V/W$. In general the quotient of a complete locally convex vector space need not be complete. However, this is the case for a *Frechet* space, i.e., a countably seminormed complete locally convex vector space (see [25]). If $V$ is an operator space, we say that it is *Frechet* if it is Frechet as a locally convex space. It follows that each space $M_n(V)$ is Frechet, and if $W$ is a closed subspace of $V$, $M_n(W)$ is closed in $M_n(V)$, and (33) determines a Frechet operator space $V/W$.

## 6. Projective limits and tensor products

The development of projective limits and tensor products for operator spaces closely parallels that for locally convex spaces. After recalling the classical definition, we continue with the development for local operator spaces. We will need the classical tensor product theory as well, but this can be deduced from the operator case, or found in [23] and [25] .

Let us suppose that we are given a vector space $V$, a family of locally convex spaces $V_\gamma$, and linear mappings $\pi_\gamma : V \to V_\gamma$. We associate with each $\gamma \in \Gamma$ and $\rho \in \mathfrak{N}(V_\gamma)$ a seminorm $\sigma$ on $V$ by letting $\sigma_n = \rho_n \circ (\pi_\gamma)_n$. Letting $\gamma$ vary, the resulting collection $\mathfrak{N}(V)$ of seminorms determines the *projective locally convex topology* on $V$ determined by the $\pi_\gamma$, and we write

$$V = \varprojlim \{V_\gamma, \pi_\gamma\} = \varprojlim V_\gamma. \tag{34}$$

If $V_\gamma$ is a locally convex subspace of a locally convex space $W_\gamma$, then it is trivial that $V = \varprojlim W_\gamma$. In this paper we will be only interested in projective limits of Banach spaces. If $V$ is a locally convex space, then we have

$$V = \varprojlim \bar{V}_\rho$$

(see (14)), i.e., $V$ is a projective limit of Banach spaces [23]. It is customary to take the completions in order to simplify the discussion of nuclearity (see the following section). It should be noted that even if $V$ is complete, this need not be the case for $V_\rho$. Given another locally convex space $W = \varprojlim \bar{W}_\sigma$, we have that a linear mapping $\varphi : V \to W$ is continuous if and only if for each $\sigma \in \mathfrak{N}(W)$, there is a $\rho \in \mathfrak{N}(V)$ and a constant $k > 0$ for which

$$\|\theta_\sigma(\varphi(v))\| \leq k \|\pi_\rho(v)\|,$$

or equivalently, we have a commutative diagram

$$\begin{array}{ccc} V & \xrightarrow{\varphi} & W \\ \downarrow & & \downarrow \\ \bar{V}_\rho & \xrightarrow[\varphi_{\sigma,\rho}]{} & \bar{W}_\sigma \end{array} \tag{35}$$

where $\varphi_{\sigma,\rho}$ is a contraction.

We may similarly define the *projective operator space topology* on $V$ determined by a family of local operator spaces $V_\gamma$, $(\gamma \in \Gamma)$, and linear mappings $\pi_\gamma : V \to V_\gamma$ by using the collecton of operator seminorms $\sigma_n = \rho_n \circ (\pi_\gamma)_n$ and we write

$$V = \operatorname{op}\varprojlim\{V_\gamma, \pi_\gamma\} = \operatorname{op}\varprojlim V_\gamma. \tag{36}$$

The corresponding matrix topology on $\mathfrak{m}(V)$ is just the usual projective limit

$$\mathfrak{m}(V) = \varprojlim \{\mathfrak{m}(V_\gamma), (\pi_\gamma)_\infty\}.$$

Once again we have that if $V$ is an arbitrary local operator space, then

$$V = \operatorname{op}\varprojlim\{\bar{V}_\rho, \pi_\rho : \rho \in \mathfrak{N}_{op}(V)\}. \tag{37}$$

where $\bar{V}_\rho$ is the completion of $V_\rho$. We recall that the completion of an operator space may be obtained by completing the underlying normed space, and then observing that each matrix space is then automatically complete (see (15)). Given $\rho, \sigma \in \mathfrak{N}_{op}(V)$ with $\rho \leq \sigma$, we have a corresponding complete contraction

$$\pi_{\rho,\sigma} : \bar{V}_\sigma \to \bar{V}_\rho$$

determined by

$$\pi_{\rho,\sigma}(\pi_\sigma(v)) = \pi_\rho(v)$$

for $v \in V$.

Given $V = \operatorname{op}\varprojlim\{V_\gamma, \pi_\gamma\}$ and $W = \operatorname{op}\varprojlim\{W_\delta, \theta_\delta\}$ with $V_\gamma$ and $W_\delta$ operator spaces, we have that a linear mapping $\varphi : V \to W$ is matrix continuous if and only if for each $\delta$ there exists a $\gamma$ and a constant $k > 0$ for which

$$\|(\theta_\delta)_n(\varphi_n(v))\| \leq k \|(\pi_\gamma)_n(v)\|,$$

or equivalently, we have (35), where $\varphi_{\sigma,\rho}$ is a completely bounded mapping of complete operator spaces.

It is important to note that we are considering the projective limit of *topologies* on $\mathfrak{m}(V)$, and not the projective limit of the underlying vector spaces $\mathfrak{m}(V_\gamma)$. To illustrate how one can use this fact, let us suppose that $V$ is a finite dimensional locally convex space. It is easy to see that we may choose a generating collection of continuous *norms* $\rho_\mu$. We have from ([5], Lemma 2.3) that if $V$ is a finite dimensional operator space, then any linear mapping of $V$ into another operator space is completely bounded. It follows that all operator norms determine the same matrix topology on $\mathfrak{m}(V)$. Thus the limit topology is determined by any one of the corresponding norms (we are indebted to S. Winkler for this observation), and we conclude that there is only one local operator structure on a finite dimensional vector space.

If we are given a subspace $W$ of $V$, we have that the relative matrix topology on $\mathfrak{m}(V)$ is determined by the restriction of the seminorms $\|\pi_\nu(\cdot)\|$, i.e., we have that

$$W = \underset{\leftarrow}{\mathrm{oplim}}\{\bar{W}_\nu, \tilde{\pi}_\nu\}$$

where $W_\nu = \pi_\nu(W)$ and $\tilde{\pi}_\nu = \pi_\nu|_W$.

The following will play an important role in the next section.

**Proposition 6.1.** *If $V$ is a locally convex space, then*

$$\min V = \underset{\leftarrow}{\mathrm{oplim}}\left\{\min \bar{V}_\rho : \rho \in \mathfrak{N}(V)\right\},$$
$$\max V = \underset{\leftarrow}{\mathrm{oplim}}\left\{\max \bar{V}_\rho : \rho \in \mathfrak{N}(V)\right\}.$$

*Proof.* Given a continuous seminorm $\rho$ on $V$, it suffices to prove that $\bar{V}_{\hat{\rho}} = \max \bar{V}_\rho$ and $\bar{V}_{\check{\rho}} = \min \bar{V}_\rho$. For the second we observe that the adjoint of the quotient mapping $V \to \bar{V}_\rho$ maps the unit ball of $V_\rho^*$ onto the polar $(B^\rho)^\circ$ of the unit set for $\rho$. Thus for any $v \in M_n(V)$ we have that

$$\begin{aligned}
\check{\rho}_n(v) &= \sup\{\|\langle\langle v, g\rangle\rangle\| : g \in (B^\rho)^\circ\} \\
&= \sup\{\|\langle\langle \pi_{\rho,n}(v), f\rangle\rangle\| : f \in V^*, \|f\| \leq 1\} \\
&= \|\pi_{\rho,n}(v)\|_{\min}.
\end{aligned}$$

On the other hand, the quotient mapping also determines a mapping of the unit ball of $\mathcal{B}(V_\rho, M_p)$ onto $(B^\rho)^{\circledcirc}_p$, from which we conclude that

$$\begin{aligned}
\hat{\rho}_n(v) &= \sup\left\{\|\langle\langle v, g\rangle\rangle\| : g \in (B^\rho)^{\circledcirc}_p, p \in \mathbb{N}\right\} \\
&= \sup\{\|\langle\langle \pi_{\rho,n}(v), f\rangle\rangle\| : f \in \mathcal{B}(V_\rho, M_p), \|f\| \leq 1\} \\
&= \|\pi_{\rho,n}(v)\|_{\max}
\end{aligned}$$

(see (31) and (32)). □

Turning to tensor products, we assume that the reader is familiar with the properties of the (complete) projective and injective tensor products $V\hat{\otimes}W$ and $V\check{\otimes}W$ of normed spaces $V$ and $W$. We denote the analogous operator space tensor products for operator spaces $V$ and $W$ by $V\hat{\otimes}_{op}W$ and $V\check{\otimes}_{op}W$. Motivated by the situation for locally convex spaces, we may use projective limits to define these tensor products for local operator spaces.

Given local operator spaces $V$ and $W$, and continuous operator seminorms $\mu \in \mathfrak{N}_{op}(V)$ and $\nu \in \mathfrak{N}_{op}(W)$ we have corresponding linear mappings

$$\pi_\mu \otimes \pi_\nu : V \otimes W \to \bar{V}_\mu \hat{\otimes}_{op} \bar{W}_\nu$$

and

$$\pi_\mu \otimes \pi_\nu : V \otimes W \to \bar{V}_\mu \check{\otimes}_{op} \bar{W}_\nu.$$

We define the (*incomplete*) *projective* and *injective tensor products* $V \otimes_\wedge^{op} W$, and $V \otimes_\vee^{op} W$, respectively, to be the vector space $V \otimes W$ together with the corresponding limit matrix topologies, i.e., we let

$$V \otimes_\wedge^{op} W = \underset{\leftarrow}{\operatorname{oplim}} \bar{V}_\mu \hat{\otimes}_{op} \bar{W}_\nu$$

and

$$V \otimes_\vee^{op} W = \underset{\leftarrow}{\operatorname{oplim}} \bar{V}_\mu \check{\otimes}_{op} \bar{W}_\nu .$$

where the limits are taken over $\mu \in \mathfrak{N}_{op}(V)$ and $\nu \in \mathfrak{N}_{op}(W)$. We define the (*complete*) *projective and injective operator tensor products* $V \hat{\otimes}_{op} W$ and $V \check{\otimes}_{op} W$ to be the completion of these spaces. It is a simple matter to verify that one obtains the same spaces if one uses any generating families of operator norms $\mathfrak{R}$ and $\mathfrak{S}$ for $V$ and $W$, respectively.

We may also use projective limits to define the Haagerup tensor product $V \overset{h}{\otimes} W$ for local operator spaces. Alternatively we can appeal to a result of B. Johnson to the effect that the Haagerup tensor product has a natural interpretation in terms of module tensor products (see [6]). Since we shall not be using this tensor product below, we have put off further discussion to a subsequent paper.

Given local operator spaces $V$,$W$, and $X$, we say that a bilinear mapping $\varphi : V \times W \to X$ is *jointly matrix continuous*, if the corresponding mapping

$$\varphi_{\infty,\infty} : \mathfrak{m}(V) \times \mathfrak{m}(W) \to \mathfrak{m}_{\infty \times \infty}(X)$$

is jointly continuous in the matrix topologies. Equivalently, for each $\rho \in \mathfrak{N}(X)$ we have that there exist $\mu \in \mathfrak{N}_{op}(V)$ and $\nu \in \mathfrak{N}_{op}(W)$ and a constant $k > 0$ such that

$$\rho_{m \times n}(\varphi_{m,n}(v, w)) \leq k \mu_m(v) \nu_n(w)$$

for all $v \in M_m(V)$ and $w \in M_n(W)$. Letting $\tilde{\varphi} : V \otimes W \to X$ be the linear mapping determined by $\varphi$, it follows that we have a commutative diagram

$$\begin{array}{ccc} V \otimes W & \xrightarrow{\tilde{\varphi}} & X \\ \downarrow & & \downarrow \\ \bar{V}_\mu \hat{\otimes}_{op} \bar{W}_\nu & \xrightarrow[\tilde{\varphi}_{\mu,\nu;\rho}]{} & \bar{X}_\rho \end{array}$$

where the completely bounded linear mappings $\tilde{\varphi}_{\mu,\nu;\rho}$ are determined by the corresponding bilinear functions

$$\bar{V}_\mu \times \bar{W}_\nu \to \bar{X}_\rho .$$

It follows that $\tilde{\varphi} : V \otimes_\wedge W \to X$ is matrix continuous. Letting $\mathcal{C}_{op}(V \times W, X)$ be the vector space of all matrix continuous bilinear mappings $\varphi : V \times W \to X$, it follows that we have a natural vector space isomorphism

$$\mathcal{C}_{op}(V \times W, X) \cong \mathcal{C}_{op}(V \otimes_\wedge^{op} W, X).$$

If $W$ is an operator space, and we let $W^*$ have the dual operator space structure (see (2)) we have the linear isomorphism

$$(V \otimes_\wedge^{op} W)' \cong \mathcal{C}_{op}(V, W^*). \tag{38}$$

To see this, let us suppose that we are given a bilinear mapping $F : V \times W \to \mathbb{C}$, and that we define

$$\varphi_F : V \to W^*$$

by letting $\varphi_F(v)(w) = F(v, w)$. We have that $\varphi_F$ is matrix continuous if and only if for each $\rho \in \mathfrak{N}_{op}(V)$ there is a $k_\rho > 0$ with

$$\|F_{m,n}(v, w)\| = \|(\varphi_F)_m(v)_n(w)\| \le k_\rho \rho_m(v)\, \|w\|$$

for all $v \in M_m(V), w \in M_n(W), m, n \in \mathbb{N}$. Since the operator norms determine the operator space topology for $V$, these inequalities correspond exactly to the condition that $F \in \mathcal{C}_{op}(V \times W, \mathbb{C})$.

Given local operator space inclusions $V_0 \subseteq V$ and $W_0 \subseteq W$ (see §5), we claim that we a have natural inclusion

$$V_0 \check{\otimes}_{op} W_0 \subseteq V \check{\otimes}_{op} W. \tag{39}$$

To see this, we observe that these inclusions determine a continuous bijection of $V_0 \otimes^{op}_{\vee} W_0$ onto a subspace $E$ of $V \otimes^{op}_{\vee} W$. Given $\mu \in \mathfrak{N}_{op}(V)$ and $\nu \in \mathfrak{N}_{op}(W)$ with restrictions $\mu_0 \in \mathfrak{N}_{op}(V_0)$ and $\nu_0 \in \mathfrak{N}(W_0)$, the completely isometric inclusions $(V_0)_{\mu_0} \hookrightarrow V_\mu$ and $(W_0)_{\nu_0} \hookrightarrow W_\nu$determine a complete isometry of $\bar{V}_{0\mu} \check{\otimes}_{op} \bar{W}_{0\nu}$ onto a subspace $E_{\mu,\nu}$ of $\bar{V}_{0\mu} \check{\otimes}_{op} \bar{W}_{0\nu}$. Thus the relation (39) is apparent from the diagram

$$\begin{array}{ccccc} V_0 \otimes^{op}_{\vee} W_0 & \rightleftarrows & E & \subseteq & V \otimes^{op}_{\vee} W \\ \downarrow & & \downarrow & & \downarrow \\ \bar{V}_{0\mu} \check{\otimes}_{op} \bar{W}_{0\nu} & \rightleftarrows & E_{\mu,\nu} & \subseteq & \bar{V}_{\mu} \check{\otimes}_{op} \bar{W}_{\nu} \end{array}.$$

If $V$ is a Banach space, we have that

$$M_n(\min V) = M_n \check{\otimes} V = M_n \otimes_{\vee} V,$$

and thus

$$\mathfrak{m}(\min V) = \mathfrak{m} \otimes_{\vee} V.$$

On the other hand, we have that the vector space identification

$$M_n(\max V) = M_n \otimes V$$

determines a cross norm on $M_n \otimes V$ since if we are given $\alpha \in M_n$ and $v \in V$, we have that

$$\begin{aligned} \|\alpha \otimes v\|_{\max} &= \sup \{\|\langle\langle \alpha \otimes v, f\rangle\rangle\| : f \in \mathcal{B}(V, M_p), \|f\| \le 1, p \in \mathbb{N}\} \\ &= \sup \{\|(id \otimes f)(\alpha \otimes v)\| : f \in \mathcal{B}(V, M_p), \|f\| \le 1, p \in \mathbb{N}\} \\ &= \sup \{\|\alpha \otimes f(v)\| : f \in \mathcal{B}(V, M_p), \|f\| \le 1, p \in \mathbb{N}\} \\ &= \sup \{\|\alpha\|\, \|f(v)\| : f \in \mathcal{B}(V, M_p), \|f\| \le 1, p \in \mathbb{N}\} \\ &= \|\alpha\|\, \|v\|\,. \end{aligned}$$

It follows that we have a contraction

$$M_n \otimes_{\wedge} V \to M_n(\max V).$$

Owing to the fact that we have a contraction $\theta : \mathfrak{m} \to M_n$ for which

$$M_n \hookrightarrow \mathfrak{m} \xrightarrow{\theta} M_n$$

is the identity mapping, we have that $\mathfrak{m} \otimes_\wedge V$ is just the union of the normed spaces $M_n \otimes_\wedge V$. Since $\mathfrak{m}(\max V)$ is the union of the normed spaces $M_n(\max V)$, we conclude from the diagram

$$\begin{array}{ccc} M_n \otimes_\wedge V & \to & M_n(\max V) \\ \downarrow & & \downarrow \\ \mathfrak{m} \otimes_\wedge V & \to & \mathfrak{m}(\max V) \end{array}$$

that the natural mapping

$$\mathfrak{m} \otimes_\wedge V \to \mathfrak{m}(\max V)$$

is a contraction.

If $V$ is any locally convex space, we conclude that we have the homeomorphic identifications

$$\mathfrak{m}(\min V) = \varprojlim \mathfrak{m}(\min V_\rho) = \varprojlim \mathfrak{m} \otimes_\vee V_\rho = \mathfrak{m} \otimes_\vee V. \tag{40}$$

On the other hand, we have from the diagrams

$$\begin{array}{ccc} \mathfrak{m} \otimes_\wedge V & \to & \mathfrak{m}(\max V) \\ \downarrow & & \downarrow \\ \mathfrak{m} \otimes_\wedge \bar{V}_\rho & \to & \mathfrak{m}(\max \bar{V}_\rho) \end{array} \tag{41}$$

that the top row is a continuous mapping.

The bottom row of the diagram

$$\begin{array}{ccc} V \otimes_\wedge^{op} W & \xrightarrow{id} & W \otimes_\vee^{op} W \\ \Big\downarrow & & \Big\downarrow \\ \bar{V}_\mu \hat{\otimes}_{op} \bar{W}_\nu & \longrightarrow & \bar{V}_\mu \check{\otimes}_{op} \bar{W}_\nu \end{array}$$

is completely contractive and thus we have the *canonical* operator continuous mapping

$$\Phi_{op} : V \hat{\otimes}_{op} W \to V \check{\otimes}_{op} W. \tag{42}$$

## 7. Nuclear mappings and spaces

Given Banach spaces $V$ and $W$, a linear mapping $\varphi : V \to W$ is said to be *nuclear* if it is in the image of the canonical mapping

$$V^* \hat{\otimes} W \xrightarrow{\Phi} V^* \check{\otimes} W \subseteq \mathcal{B}(V, W).$$

The space $N(V, W)$ of *nuclear mappings* $\varphi : V \to W$ is defined to be the image of $\Phi$ together with the quotient norm $\|\varphi\|_{nuc}$ determined by

$$N(V, W) \cong \frac{V^* \hat{\otimes} W}{\ker \Phi}.$$

We have that a mapping $\varphi : V \to W$ satisfies $\|\varphi\|_{nuc} < 1$ if and only if we have a commutative diagram

$$\begin{array}{ccc} l_\infty & \xrightarrow{\theta_\lambda} & l_1 \\ \sigma \Big\uparrow & & \Big\downarrow \tau \\ V & \xrightarrow[\varphi]{} & W \end{array}$$

where the column mappings are contractions, and $\theta_\lambda$ is defined for a $\lambda \in l_1$ of norm $< 1$, by

$$\theta_\lambda(a_i) = (\lambda_i a_i)$$

Perturbing these mappings, we may replace $l_\infty$ in this diagram by $c_0$. A locally convex space $V$ is said to be *nuclear* if for each continuous seminorm $\rho$ on $V$, there exists a continuous seminorm $\sigma \geq \rho$ on $V$ for which the corresponding mapping $\bar{V}_\sigma \to \bar{V}_\rho$ is nuclear. We have (see [25]):

**Theorem 7.1.** *Suppose that $V$ is a locally convex space. Then the following are equivalent:*

a) *$V$ is nuclear*
b) *For all locally convex spaces $W$, we have that the mapping $V \otimes_\wedge W \to V \otimes_\vee W$ is a homeomorphism, and thus $V \hat{\otimes} W = V \check{\otimes} W$.*

Given operator spaces $V$ and $W$ we have a corresponding diagram

$$V^* \hat{\otimes}_{op} W \xrightarrow{\Phi_{op}} V^* \check{\otimes}_{op} W \subseteq \mathcal{CB}(V, W).$$

We define the space $N_{op}(V, W)$ of *matrix nuclear mappings* $\varphi : V \to W$ to be the range of $\Phi_{op}$ together with the quotient norm $\|\varphi\|^{op}_{nuc}$ determined by

$$N_{op}(V, W) \cong \frac{V^* \hat{\otimes}_{op} W}{\ker \Phi_{op}}$$

(see [9] for the properties of these mappings). We say that a mapping $\varphi : V \to W$ is a *proper matrix nuclear contraction* if it satisfies $\|\varphi\|^{op}_{nuc} < 1$. This will be the case if and only if we have a commutative diagram

$$\begin{array}{ccc} M_\infty & \xrightarrow{\theta_{a,b}} & T_\infty \\ \sigma \uparrow & & \downarrow \tau \\ V & \xrightarrow[\varphi]{} & W \end{array} \tag{43}$$

where $\sigma$ and $\tau$ are complete contractions, and the mapping $\theta_{a,b}$ is defined for properly contractive Hilbert-Schmidt operators $a, b$ by

$$\theta_{a,b}(x) = axb.$$

It is equivalent to assume that we have commutative diagrams (43) with $M_\infty$ replaced by $K_\infty$.

Given two operator spaces $V$ and $W$, we have a natural diagram of complete contractions interrelating the scalar and operator spaces:

$$\begin{array}{ccccccc} V^* \hat{\otimes} W & \longrightarrow & V^* \hat{\otimes}_{op} W & \longrightarrow & V^* \check{\otimes}_{op} W & \longrightarrow & V^* \check{\otimes} W \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ N(V, W) & \longrightarrow & N_{op}(V, W) & \longrightarrow & \mathcal{B}_{op}(V, W) & \longrightarrow & \mathcal{B}(V, W) \end{array}$$

In particular, we conclude that if $\varphi : V \to W$ is nuclear in the classical sense, then it is operator nuclear.

**Proposition 7.1.** *Suppose that $V$ and $W$ are operator spaces. If $\varphi : V \to W$ is a proper matrix nuclear mapping with $\|\varphi\|_{nuc}^{op} < 1$, then for any operator space $X$ the mapping*

$$\varphi \otimes id : V \otimes_{\vee}^{op} X \to W \otimes_{\wedge}^{op} X$$

*is a complete contraction, and thus determines a complete contraction*

$$\varphi \otimes id : V \check{\otimes}_{op} X \to W \hat{\otimes}_{op} X.$$

*Proof.* To see this we recall that if $G \in (W \hat{\otimes}_{op} X)^*$ and $\|G\| \leq 1$, then $G$ determines a complete contraction

$$\theta_G : W \to X^*$$

by

$$\theta_G(w)(x) = G(w \otimes x)$$

(see (38)). It is evident from (43) that the composition

$$V \xrightarrow{\varphi} W \xrightarrow{\theta_G} X^*$$

is again a proper matrix nuclear contraction, and thus has the form $\pi(F)$ for some $F \in V^* \hat{\otimes}_{op} X^*$ with $\|F\| < 1$. We have that

$$\langle G, \varphi(v) \otimes w \rangle = \pi(F)(v)(w) = \langle F, v \otimes w \rangle,$$

and by linearity,

$$\langle G, (\varphi \otimes id)(u) \rangle = \langle F, u \rangle$$

for all $u \in V \otimes X$. Since the mapping

$$V \check{\otimes}_{op} W \hookrightarrow (V^* \hat{\otimes}_{op} W^*)^*$$

is completely isometric (see [2]), it follows that

$$\begin{aligned} \|\varphi \otimes id(u)\| &= \sup \{ |\langle G, (\varphi \otimes id)(u) \rangle| : \|G\| \leq 1 \} \\ &\leq \sup \{ |\langle F, u \rangle| : \|F\| \leq 1 \} \\ &\leq \|u\|_{\vee} . \end{aligned}$$

□

Given operator spaces $V$ and $W$ and matrix nuclear mappings $\varphi : V \to W$ and $\psi : X \to Y$, we have that the mapping

$$\varphi \otimes \psi : V \hat{\otimes}_{op} X \to W \hat{\otimes}_{op} Y$$

is matrix nuclear. To see this let us assume that we have the commutative diagram (43) for $\varphi$ as well as a commuative diagram

$$\begin{array}{ccc} K_\infty & \xrightarrow{\theta_{c,d}} & T_\infty \\ \sigma' \uparrow & & \downarrow \tau' \\ X & \xrightarrow[\psi]{} & Y \end{array}$$

Owing to the universal property of the norms $\hat{\otimes}_{op}$, we have that the first column in the commutative diagram

$$\begin{array}{ccc} K_\infty \check{\otimes}_{op} K_\infty = K_{\infty\times\infty} & \xrightarrow{\theta_{a\otimes c, b\otimes d}} & T_{\infty\times\infty} = T_\infty \hat{\otimes}_{op} T_\infty \\ {\scriptstyle \sigma\otimes\sigma'}\uparrow & & \downarrow{\scriptstyle \tau\otimes\tau'} \\ V \hat{\otimes}_{op} X & \xrightarrow[\varphi\otimes\psi]{} & W \hat{\otimes}_{op} Y \end{array}$$

is completely contractive. Since it is evident that $a\otimes c$ and $b\otimes d$ are Hilbert-Schmidt, we are done.

Given local operator spaces $V$ and $W$, we say that a linear mapping $\varphi : V \to W$ is *matrix* or *operator nuclear* if there exists a commutative diagram

$$\begin{array}{ccc} E & \xrightarrow{\psi} & F \\ {\scriptstyle \sigma}\uparrow & & \downarrow{\scriptstyle \tau} \\ V & \xrightarrow{\varphi} & W \end{array}$$

where $E$ and $F$ are *complete operator spaces*, $\sigma$ and $\tau$ are continuous, and $\psi$ is nuclear.

We say that a local operator space $V$ is *matrix* or *operator nuclear* if for any continuous operator seminorm $\rho \in \mathfrak{N}_{op}(V)$, we have that there is a $\sigma \in \mathfrak{N}_{op}(V)$ with $\sigma \geq \rho$ and for which the corresponding mapping

$$\pi_{\rho,\sigma} : \bar{V}_\sigma \to \bar{V}_\rho \tag{44}$$

is matrix nuclear.

**Theorem 7.2.** *If $V$ is local operator space, then the following are equivalent:*

a) *$V$ is a matrix nuclear local operator space,*
b) *$V$ is a nuclear locally convex space.*

*Proof.* Let us suppose that $V$ is matrix nuclear. Then given $\rho \in \mathfrak{N}_{op}(V)$ we may select $\sigma, \tau \in \mathfrak{N}_{op}(V)$ with $\tau \geq \sigma \geq \rho$ for which the mappings

$$\bar{V}_\tau \to \bar{V}_\sigma \to \bar{V}_\rho$$

are matrix nuclear. But it was shown in [10], Th. 4.3, that any matrix nuclear mapping has factorizations through both row and column Hilbert spaces (see §1). Thus we obtain a diagram of completely contractive mappings

$$\begin{array}{ccccccc} & & H_c & \longrightarrow & K_r & & \\ & \nearrow & & \searrow \quad \nearrow & & \searrow & \\ \bar{V}_\tau & & \longrightarrow & \bar{V}_\sigma & \longrightarrow & & \bar{V}_\rho \end{array}$$

with $H$ and $K$ Hilbert spaces, and where the top row is just the composition $H_c \to \bar{V}_\sigma \to K_r$. But any such completely contractive mapping from $H_c \to K_r$ must be Hilbert Schmidt (see [8], Cor. 4.5), and the same therefore applies to the composition $\bar{V}_\tau \to \bar{V}_\rho$. We may next use the fact that a composition of Hilbert-Schmidt operators is a trace class operator, and the trace class operators coincide with the nuclear operators between Hilbert spaces. Thus if we select a larger seminorm $\lambda$, we obtain the Hilbert spaces $F$ and $G$, and a diagram of mappings

$$\begin{array}{ccccccccc} F & \rightarrow & G & & \longrightarrow & & H & \rightarrow & K \\ \uparrow & & & \searrow & & \nearrow & & & \downarrow \\ \bar{V}_\lambda & & \longrightarrow & & \bar{V}_\tau & & \longrightarrow & & \bar{V}_\rho \end{array}$$

where the mapping $G \to H$ is just the obvious composition. It follows that we obtain diagrams

$$\begin{array}{ccc} F & \xrightarrow{\theta} & K \\ \Big\uparrow & & \Big\downarrow \\ \bar{V}_\lambda & \longrightarrow & \bar{V}_\rho \end{array}$$

where the mappings $\theta$ are nuclear, and thus $V$ is nuclear as a locally convex space.

Conversely suppose that $V$ is a nuclear locally convex space. Then given $\rho \in \mathfrak{N}_{op}(V)$, we may find a $\sigma \in \mathfrak{N}(V)$ such that $\sigma \geq \rho$, and the corresponding mapping $\bar{V}_\sigma \to \bar{V}_\rho$ is nuclear. But the operator seminorms are assumed to define the topology on $V$, hence we may choose $\tau \in \mathfrak{N}_{op}(V)$ such that $\tau \geq \sigma$. It follows that the composition $\bar{V}_\tau \to \bar{V}_\rho$ is nuclear and from our earlier remarks, it must also be operator nuclear. □

In this regard, we note that $\mathfrak{m}(V) = \mathfrak{m} \otimes_\vee V$ is never nuclear as a locally convex space. To see this we simply observe that if we fix an element $v_0 \in V$, we have that we have the homeomorphic inclusion

$$\mathfrak{m} \cong \mathfrak{m} \otimes_\vee \mathbb{C}v_0 \hookrightarrow \mathfrak{m} \otimes_\vee V,$$

and we have that a subspace of a nuclear space is nuclear. Thus if $\mathfrak{m}(V)$ is nuclear, then so is the infinite dimensional normed space $\mathfrak{m}$, which is a contradiction (see [25], pp. 100-103).

**Theorem 7.3.** *If $V$ is a nuclear locally convex space, then it has precisely one quantization.*

*Proof.* It suffices to prove that $\max V = \min V$. But we have a diagram of topological identifications and continuous mappings

$$\mathfrak{m}(\min V) = \mathfrak{m} \otimes_\vee V = \mathfrak{m} \otimes_\wedge V \to \mathfrak{m}(\max V) \to \mathfrak{m}(\min V).$$

for which the composition is the identity mapping. The first equality follows from (40), the second from the fact that $V$ is a nuclear locally convex space. The first mapping is explained in (41). The final mapping is continuous since the corresponding result is trivial for operator spaces, and we have the commutative diagrams

$$\begin{array}{ccc} \max V & \rightarrow & \min V \\ \downarrow & & \downarrow \\ \max \bar{V}_\rho & \rightarrow & \min \bar{V}_\rho \end{array}$$

Thus we have the desired equality. □

**Theorem 7.4.** *If $V$ is a matrix nuclear local operator space, then for any local operator space $W$ we have that*

$$V \hat{\otimes}_{op} W = V \check{\otimes}_{op} W. \tag{45}$$

*Proof.* We have already seen from (42) that we have a matrix continuous mapping from the first space to the second. On the other hand given $\rho \in \mathfrak{N}_{op}(V)$, let us select $\rho_1 \geq \rho$ for which the mapping $\bar{V}_{\rho_1} \to \bar{V}_\rho$ is matrix nuclear. Then for any $\sigma \in \mathfrak{N}_{op}(W)$, we have from Proposition 7.1, that the bottom row of the diagram

$$\begin{array}{ccc} V \otimes_\vee^{op} W & \longrightarrow & V \otimes_\wedge^{op} W \\ \downarrow & & \downarrow \\ \bar{V}_{\rho_1} \check{\otimes}_{op} \bar{W}_\sigma & \longrightarrow & \bar{V}_\rho \hat{\otimes}_{op} \bar{W}_\sigma \end{array}$$

is completely bounded, and thus the top row is matrix continuous. □

In contrast to the situation for locally convex spaces, it is not evident that the property (45) characterizes matrix nuclearity for $V$. The difficulty stems from the fact that operator spaces need not satisfy the analogue of local reflexivity [5]. However, if we take the latter condition into consideration, we obtain a completely satisfactory result.

Given operator space $V$ and $W$, we say that a mapping $\varphi : V \to W$ is *operator integral*, if we have approximately commutative diagrams

$$\begin{array}{ccc} M_n & \xrightarrow{\theta_{c,d}} & T_n \\ \sigma \uparrow & & \downarrow \tau \\ V & \underset{\varphi}{\longrightarrow} & W \end{array}$$

(see [9]). As in the classical case, any matrix nuclear mapping is operator integral. On the other hand, the composition of two operator integral mappings is matrix nuclear. It follows from the latter fact that a local operator space $V$ is nuclear if and only if we have diagrams (44) with $\pi_{\rho,\sigma}$ integral. We let $I_{op}(V, W)$ denote the linear space of integral mappings between operator spaces $V$ to $W$. We will have no need to define the corresponding notion for local operator spaces.

An operator space $W$ is *locally operator reflexive* if for each finite dimensional operator space $E$ we have that any completely contractive mapping $\varphi : E \to W^{**}$ can be approximated in the point-norm topology by completely contractive mappings $\varphi : E \to W$. Equivalently, we have that for any operator space $V$, the natural mapping

$$I_{op}(V, W^*) \to (V \check{\otimes}_{op} W)^* \tag{46}$$

is a completely isometric bijection (see [9], Th. 3.6).

**Proposition 7.2.** *Given a local operator space $V$, we have that $V$ is matrix nuclear if and only if we have*

a) *$V_\rho^*$ is locally reflexive for a generating family of continuous operator seminorms $\rho \in \mathfrak{N}_{op}(V)$, and*
b) *for all operator spaces $W$, $V \hat{\otimes}_{op} W = V \check{\otimes}_{op} W$.*

*Proof.* Let us assume that we are given $\rho \in \mathfrak{N}_{op}(V)$. From (38) the mapping $\pi_\rho : V \to \bar{V}_\rho$ determines a continuous linear functional

$$F \in (V \hat{\otimes}_{op} V_\rho^*)',$$

where $F(v \otimes f) = f(\pi_\rho(v))$. From b), we have $F \in (V\check{\otimes}_{op}V_\rho^*)'$. Since by definition,

$$V\check{\otimes}_{op}V_\rho^* = \varprojlim_\sigma V_\sigma\check{\otimes}_{op}V_\rho^*,$$

there is a $\sigma \in \mathfrak{N}_{op}(V)$ and an element

$$\bar{F} \in (V_\sigma\check{\otimes}_{op}V_\rho^*)^*$$

with

$$F(v \otimes f) = \bar{F}(\pi_\sigma(v) \otimes f).$$

for $f \in V_\rho^*$. From a) and (46)we may assume that

$$(V_\sigma\check{\otimes}_{op}V_\rho^*)^* = I_{op}(V_\sigma, V_\rho^{**}).$$

We may also assume that $\sigma \geq \rho$, and thus that we have a corresponding mapping

$$\pi_{\rho,\sigma} : \bar{V}_\sigma \to \bar{V}_\rho$$

for which $\pi_\rho = \pi_{\rho,\sigma} \circ \pi_\sigma$. If $v \in V$, then

$$\begin{aligned} \bar{F}(\pi_\sigma(v) \otimes f) &= F(v \otimes f) \\ &= f(\pi_\rho(v)) \\ &= f(\pi_{\rho,\sigma}(\pi_\sigma(v)), \end{aligned}$$

and thus $\bar{F}$ is determined by the mapping $\pi_{\rho,\sigma} : \bar{V}_\sigma \to \bar{V}_\rho$. Owing to the fact that the natural inclusion

$$I_{op}(V_\sigma, V_\rho) \hookrightarrow I_{op}(V_\sigma, V_\rho^{**})$$

is completely isometric (see [9]) we conclude that the connecting mapping $\pi_{\rho,\sigma}$ is operator integral. From our earlier remarks, we conclude that $V$ is matrix nuclear.

Conversely if $V$ is matrix nuclear, then it is evident from the proof of Theorem 7.2 we may assume that $V$ is a projective limit of column Hilbertspaces. Since reflexive operator spaces are obviously locally reflexive, we have condition a). Condition b) follows from Theorem 7.4 □

**Proposition 7.3.** *If $V$ and $W$ are matrix nuclear spaces, then the same is true for $V\hat{\otimes}_{op}W$.*

*Proof.* This is immediate from the fact that if we choose $\rho_1 \geq \rho$ and $\sigma_1 \geq \sigma$ with the corresponding mappings $\bar{V}_{\rho_1} \to \bar{V}_\rho$ and $\bar{W}_{\sigma_1} \to \bar{W}_\sigma$ matrix nuclear, it follows that

$$\bar{V}_{\rho_1}\hat{\otimes}_{op}\bar{W}_{\sigma_1} \to \bar{V}_\rho\hat{\otimes}_{op}\bar{W}_\sigma$$

is matrix nuclear. □

## 8. Direct limits and operator bounded sets

Given a vector space $V$ and a family of normed spaces $E_\gamma$ together with mappings $\psi_\gamma : E_\gamma \to V$, the *direct limit topology* on $V$ is defined to be the finest locally convex topology on $V$ for which these mappings are all continuous. We write

$$V = \varinjlim\{E_\gamma, \psi_\gamma\}$$

to indicate $V$ with this topology. Given a locally convex topology on $V$, this relation holds if and only if it has the following property: a linear mapping $\varphi: V \to W$, with $W$ locally convex, is continuous if and only if each of the compositions

$$E_\gamma \to V \to W$$

is continuous (see [23], p. 79).

If the $E_\gamma$ are operator spaces, then we have corresponding mappings

$$(\psi_\gamma)_\infty : \mathfrak{m}(E_\gamma) \to \mathfrak{m}(V),$$

and we define the *direct limit matrix topology on* $V$ to be the corresponding direct limit topology on $\mathfrak{m}(V)$, i.e.,

$$\mathfrak{m}(V) = \varinjlim \mathfrak{m}(E_\gamma).$$

We use the notation

$$V = \operatorname{op}\varinjlim\{E_\gamma, \psi_\gamma\}$$

to indicate $V$ with this matrix topology. Again it is evident that we have that if $V$ is a local operator space, this relation will be true if and only if $V$ has the following property: a linear mapping $\varphi: V \to W$, with $W$ a local operator space, is matrix continuous if and only if each of the compositions

$$E_\gamma \to V \to W$$

is matrix continuous.

A subset $B$ of a locally convex space is said to be *bounded* if it is absorbed by each neighborhood $N$ of 0, i.e., we have that $cB \subseteq N$ for some constant $c > 0$. Equivalently, each continuous seminorm $\nu \in \mathfrak{N}(V)$ is bounded on $B$. If $B$ is a bounded closed absolutely convex set with matrix gauge $p = p^B$, we have that for any $\rho \in \mathfrak{N}(V)$ there is a constant $k > 0$ such that $p \le k\rho$ . It follows that $p$ is faithful, and it defines a norm $p$ is a norm on ${}_pV$ (see (13)) for which the inclusion mapping

$$\iota_p : {}_pV \hookrightarrow V$$

is continuous. We let $\mathfrak{B}(V)$ denote the set of all gauges of bounded closed absolutely convex sets in $V$. Given locally convex spaces $V$ and $W$, we say that a linear mapping $\varphi : V \to W$ is *bounded* if it maps bounded sets into bounded sets. We let $\mathcal{B}(V, W)$ denote the linear space of bounded linear mappings. Any continuous linear mapping $\varphi : V \to W$ is bounded, hence we have the linear injection

$$\mathcal{C}(V, W) \hookrightarrow \mathcal{B}(V, W).$$

A subset $P$ of a locally convex space $V$ is said to be *bornivorous* (see [18]) if it absorbs arbitrary bounded sets, i.e., for any bounded set $B$ there is a $k > 0$ with $kB \subseteq P$. Any neighborhood of the origin is obviously bornivorous. A locally convex space $V$ is said to *bornological* if conversely, any bornivorous set is a neighborhood of 0. Any countably generated locally convex space is bornological [23], p.82. The following result may be found in [18], p. 220 and [25].

**Proposition 8.1.** *Suppose that $V$ is a locally convex space. Then the following are equivalent:*

a) *$V$ is bornological.*

b) *If $W$ is locally convex and $\varphi : V \to W$ is bounded, then it is continuous, i.e., we have that $\mathcal{C}(V, W) = \mathcal{B}(V, W)$.*

c) $V = \varinjlim \{{}_pV : p \in \mathfrak{B}(V)\}$.

We let $\mathfrak{B}_{op}(V)$ denote the gauges of bounded $\mathfrak{m}$-convex sets in $\mathfrak{m}(V)$. Any bounded set in $\mathfrak{m}(V)$ is contained in a bounded $\mathfrak{m}$-convex set. To see this let us suppose that $\rho \in \mathfrak{N}_{op}(V)$, and that $\rho_\infty(B) \leq \alpha$. Then it is immediate from **MG1**, **MG2**, and Lemma 3.2 that if $\rho \in \mathfrak{N}_{op}(V)$, and $\rho_\infty(B) \leq \alpha$ then $\rho_\infty(h_{op}(B)) \leq \alpha$. It follows that

$$\varinjlim \{{}_{p_\infty}\mathfrak{m}(V) : p \in \mathfrak{B}_{op}(V)\} = \varinjlim \{{}_q\mathfrak{m}(V) : q \in \mathfrak{B}(\mathfrak{m}(V))\} \tag{47}$$

We say that a mapping $\varphi : V \to W$ is *matrix* (or *operator*) *bounded* if $\varphi_\infty : \mathfrak{m}(V) \to \mathfrak{m}(W)$ is bounded, and we let $\mathcal{B}_{op}(V, W)$ be the linear space of matrix bounded mappings.

**Corollary 8.1.** *Suppose that $V$ is a local operator space. Then the following are equivalent:*

a) *$\mathfrak{m}(V)$ is bornological*

b) $V = \operatorname{oplim}_{\to} \{{}_pV : p \in \mathfrak{B}_{op}(V)\}$.

*If $V$ satisfies either of these properties, then for any operator space $W$, we have that $\mathcal{C}_{op}(V, W) = \mathcal{B}_{op}(V, W)$. Furthermore, any countably generated local operator space is matrix bornological.*

*Proof.* If $V$ is matrix bornological, then we have from the proposition and (47) that

$$V = \varinjlim \{{}_q\mathfrak{m}(V) : q \in \mathfrak{B}(\mathfrak{m}(V))\} = \varinjlim \{{}_{p_\infty}\mathfrak{m}(V) : p \in \mathfrak{B}_{op}(V)\}$$

and thus we have b). The converse relation is immediate, as are the remaining two assertions. □

If $B$ is a closed bounded convex set in a locally convex space $V$, then its polar $B^\circ$ is absorbing in $V'$, since any continuous linear functional $f$ on $V$ must be bounded on $B$. Thus we have that if $p$ is the gauge of a bounded set in $V$, it is faithful, and its polar $\rho = p^\circ$ is a seminorm on $V'$. We have the obvious norm isomorphism

$$({}_pV)^* \cong (V')_\rho \tag{48}$$

The same arguments apply to the gauges of bounded $\mathfrak{m}$-convex sets in $\mathfrak{m}(V)$. We have that if $p \in \mathfrak{B}_{op}(V)$, then $\rho = p^{\circledcirc}$ is an $\mathfrak{m}$-module seminorm on $\mathfrak{m}(V')$, and in this context we have that (48) is a complete isometry of operator spaces. We shall use such seminorms in the following section to define topologies for mapping spaces.

## 9. Mapping spaces and duality

Let us suppose that we are given local operator spaces $V$ and $W$. For each operator bounded gauge $p \in \mathfrak{B}_{op}(V)$ and continuous operator seminorm $\rho \in \mathfrak{N}_{op}(W)$, we define

$$\pi_{\rho,p} : \mathcal{C}_{op}(V, W) \to \mathcal{B}_{op}(V_p, W_\rho)$$

by taking a linear mapping $\varphi \in \mathcal{C}_{op}(V, W)$ into the composition

$${}_pV \longrightarrow V \xrightarrow{\varphi} W \longrightarrow W_\rho .$$

Given a collection of operator bounded gauges $\mathfrak{S} \subseteq \mathfrak{B}(V)$. we define the $\mathfrak{S}$ *matrix topology* on $\mathcal{C}_{op}(V, W)$ by the relation

$$\mathcal{C}_{op}(V, W)_{\mathfrak{S}} = \underset{\leftarrow}{\text{oplim}}\{\mathcal{B}_{op}({}_pV, W_\rho), \pi_{p,\rho} : p \in \mathfrak{S}, \rho \in \mathfrak{N}_{op}(W)\}.$$

In particular, we let

$$V'_{\mathfrak{S}} = \mathcal{C}_{op}(V, \mathbb{C})_{\mathfrak{S}}.$$

Given a net

$$\varphi_\nu \in \mathfrak{m}(\mathcal{C}_{op}(V, W)) = \mathcal{C}_{op}(V, \mathfrak{m}(W)),$$

we have that $\varphi_\nu \to \varphi$ if and only if for each $\rho \in \mathfrak{N}_{op}(V)$ we have that the compositions

$${}_pV \hookrightarrow V \xrightarrow{\varphi_\nu} \mathfrak{m}_\infty(W) \to \mathfrak{m}_\infty(\bar{W}_\rho)$$

converge in $\mathcal{B}_{op}({}_pV, \mathfrak{m}(W_\rho))$ to

$${}_pV \hookrightarrow V \xrightarrow{\varphi} \mathfrak{m}_\infty(W) \to \mathfrak{m}_\infty(\bar{W}_\rho)$$

In particular, we have that $f_\nu \in \mathfrak{m}(V')$ converges to $f \in \mathfrak{m}(V')$ if and only if

$$\langle\langle f_\nu, v\rangle\rangle \to \langle\langle f, \nu\rangle\rangle$$

in the norm topology on $\mathfrak{m}_{\infty\times\infty}$, uniformly on the bounded unit sets $B^p \subseteq \mathfrak{m}(V)$ for $p \in \mathfrak{S}$.

We define the *strong* matrix topology on

$$V' = \mathcal{C}_{op}(V, \mathbb{C})$$

by using the collection $\beta = \mathfrak{B}_{op}(V)$ of all bounded operator norms on $V$. The matrix topology on $\mathfrak{m}(V')$ is determined by the $\mathfrak{m}$-module seminorms $\rho = p^{\circledcirc}$ with $p \in \beta$. We have that a net $f_\nu \in \mathfrak{m}(V')$ converges to a function $f \in \mathfrak{m}(V')$ in this topology if the functions $v \mapsto \langle\langle f_\nu, v\rangle\rangle$ converge uniformly to the function $v \mapsto \langle\langle f, v\rangle\rangle$ on the operator bounded sets in $\mathfrak{m}(V)$. We note that since the seminorms $\rho_1$ with $\rho \in \mathfrak{N}_{op}(V)$ determine the topology on $V$, if $B$ is a subset in $V$, then it is bounded in $V$ if and only if it is bounded in $\mathfrak{m}(V)$ (we are using the obvious inclusion $V \hookrightarrow \mathfrak{m}(V)$). Thus the strong matrix topology on $\mathfrak{m}(V')$ restricts to the usual strong topology on $V'$.

We let $V''$ denote the *strong bidual* of a local operator space $V$, i.e., $V'' = (V'_\beta)'_\beta$.

**Proposition 9.1.** *If $V$ is an operator bornological space, then the mapping $V \to V''$ is a matrix homeomorphic injection.*

*Proof.* The matrix topology on $V''$ is generated by the matrix seminorms $\sigma = p^{\circledcirc}$ with $p$ the matrix gauge of a bounded $\mathfrak{m}$-convex set $D \subseteq \mathfrak{m}(V')$. In particular, if $N$ is an open set in $\mathfrak{m}(V)$, it absorbs bounded sets $K$ in $\mathfrak{m}(V)$, and thus $N^{\circledcirc}$ is absorbed by sets of the form $K^{\circledcirc}$ in $\mathfrak{m}(V')$. Since the sets $K^{\circledcirc}$ with $K$ bounded in $\mathfrak{m}(V)$ generate the strong topology on $\mathfrak{m}(V')$, it follows that $N^{\circledcirc}$ is bounded. On the other hand, if $D$ is a bounded set in $\mathfrak{m}(V')$, then it is absorbed by sets of the form $K^{\circledcirc}$ with $K$ bounded in $V$. It follows that $D_{\circledcirc} = D^{\circledcirc} \cap \mathfrak{m}(V)$ absorbs the bounded $\mathfrak{m}$-convex sets in $\mathfrak{m}_\infty(V)$. Thus if we assume that $V$ is operator bornological, $D_{\circledcirc}$ contains a neighborhood $N$ of 0 in $\mathfrak{m}(V)$, and $(D_{\circledcirc})^{\circledcirc}$ is contained in a set of the form $N^{\circledcirc}$. Thus the strong topology on $V''$ is determined by sets of the form $N^{\circledcirc\circledcirc}$. Since we have that

$$N^{\circledcirc\circledcirc} \cap V = N \cap V,$$

it follows that the injection $V \hookrightarrow V''$ is a matrix homeomorphism. □

## 10. A nuclear quantization arising in quantum mechanics

It follows from §7 that there is a one-to-one correspondence between nuclear locally convex spaces and nuclear operator spaces. It perhaps useful to consider a very simple example that arises in mathematical physics [4], in which we have explicit operator nuclear mappings.

For any Hilbert spaces $H$ and $K$ we have the natural *isometry*

$$H_r \hat{\otimes} K_c = H_r \hat{\otimes}_{op} K_c \tag{49}$$

(see [8], Cor. 4.4c). Replacing $H$ by $H^*$, it follows that we have a commutative diagram of isometries

$$\begin{array}{cccc} & H^* \hat{\otimes} K & \cong & (H_c)^* \hat{\otimes}_{op} K_c \\ & \downarrow & & \downarrow \\ T(H,K) = & N(H,K) & \cong & N_{op}(H_c, K_c) \end{array}$$

i.e., each trace class contraction $\varphi : H \to K$ determines a matrix nuclear mapping $\varphi : H_c \to K_c$. Given an arbitrary nuclear Frechet space $V$, we may find a diagram of Hilbert spaces and nuclear mappings

$$H_1 \overset{\varphi_1}{\longleftarrow} H_2 \overset{\varphi_2}{\longleftarrow} \dots$$

with

$$V = \lim_{\leftarrow} H_k.$$

We then have corresponding nuclear operator spaces $V_c$ and $V_r$ with the same underlying locally convex topologies, and the matrix topologies

$$V_c = \operatorname{oplim}_{\leftarrow}(H_k)_c, \text{ and}$$

$$V_r = \operatorname{oplim}_{\leftarrow}(H_k)_r.$$

We let $s$ be the space of sequences $a = (a_n)_{n\in\mathbb{N}}$ which are rapidly vanishing in the sense that

$$\sup \left\{ n^k \left| a_n \right| \right\} < \infty$$

for all $k \in \mathbb{N}$, and we define a mapping

$$\pi_k : s \to l_2$$

by letting

$$\pi_k(a)_n = (1+n)^{2k} a_n.$$

We then have a diagram of spaces

$$\begin{array}{ccccc} & & & & s \\ & & & \swarrow \pi_2 & \downarrow \pi_1 \\ \dots & \overset{\theta}{\longrightarrow} & l_2 & \overset{\theta}{\longrightarrow} & l_2 \end{array},$$

where $\theta : l_2 \to l_2$ is defined by

$$\theta(a)_n = (1+n)^{-2} a_n.$$

We may write $\theta = \sigma \circ \sigma$ where the diagonal operator

$$\sigma(a)_n = (1+n)^{-1} a_n$$

is a Hilbert-Schmidt mapping since

$$\sum_{i,j} |\sigma_{i,j}|^2 = \sum_j (1+j)^{-2} < \infty.$$

It follows that $\theta$ is of trace class, and thus nuclear. Letting $s$ have the topology defined by the mappings $\pi_k$, we see that $s = \lim\limits_{\leftarrow}\{l_2, \pi_k\}$is a nuclear space. Using the terminology of [4], the (one parameter) space of *physical states* $\mathcal{T}$ is defined to be the locally convex tensor product

$$\mathcal{T} = s \hat{\otimes} s.$$

It is the dual of the algebra $\mathcal{A}$ of *observables.*

In this context we may define a corresponding local operator structure on $\mathcal{T}$ as follows. We let $s_c = \lim\limits_{\leftarrow}(l_2)_c$ and $s_r = \lim\limits_{\leftarrow}(l_2)_r$ where $(l_2)_c$ and $(l_2)_r$ are the column and row Hilbert spaces determined by $l_2$. From above we have that the mappings $\theta : (l_2)_c \to (l_2)_c$ and $\theta : (l_2)_r \to (l_2)_r$ are matrix nuclear, and thus

$$\mathcal{T}_{op} = s_r \hat{\otimes}_{op} s_c = \lim_{\leftarrow} (l_2)_r \hat{\otimes}_{op} (l_2)_c$$

is a matrix nuclear local operator space.

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