BUSEMANN POINTS OF METRIC SPACES

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Abstract. We provide a geometric condition which says precisely when the metric boundary of a proper separable metric space has no Busemann points. This result generalizes an earlier result which applied only to graph metrics.

1. Introduction

The metric boundary of a metric space, although introduced by Gromov [2] in 1978 has recently become of interest in the study of metrics on the state spaces of C*-algebras. Marc Rieffel [3] demonstrated that a length function on a discrete group naturally determines a metric on the state space of the reduced group C*-algebra via a construction due to Connes [1], however it is open as to whether these metrics determine the weak-* topology on the state space. Rieffel showed that the metric compactification occurred naturally in Conne's construction, and using this fact was able to answer the question affirmatively in the case of $\mathbb{Z}^d$ for Cayley graph metrics and metrics given by the restriction of norms on $\mathbb{R}^d$ by looking at the structure of the boundary.

The metric compactification of a metric space (X, d) is the compactification given via Gelfand's theorem when adjoining constant functions and functions of the form

$$\varphi_y(x) = d(x, y) - d(x, 0)$$

to the C*-algebra of continuous functions which vanish at infinity on X. Rieffel [3] showed that the boundary of this compactification can be completely characterized as the limits of what he called weakly-geodesic rays, but in fact in many cases the points on the metric boundary are the limits of almost-geodesic rays, and he called such boundary points Busemann points.

Rieffel raised the question as to under what conditions all points on the boundary are Busemann points, and in [4] the authors of this paper gave precise conditions for graph metrics. In this paper we generalize these results to arbitrary proper separable metric spaces.

This is of some interest since although Rieffel considered only group C*-algebras of finitely generated infinite groups, and hence the metric spaces under consideration are graph metrics of Cayley graphs of such groups, the methods he used to understand the metric boundary of $\mathbb{Z}^d$ relied heavily on embedding it into $\mathbb{R}^d$ with an appropriate norm. This approach of embedding a finitely generated metric space into some other well-understood, but not finitely generated, metric group may be more generally applicable, and in this case understanding the metric boundary of such groups may be required.

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2. Busemann Points

We follow [3, 4] in presenting the basic definitions for this paper. Let (X, d) be a complete, locally compact metric space, and let $C_\infty(X)$ be the commutative C*-algebra of continuous functions which vanish at infinity on X, and let $\mathcal{G}(X, d)$ be the C*-algebra of bounded continuous functions on X generated by $C_\infty(X)$, the constant functions, and functions of the form

$$\varphi_y(x) = d(x, y) - d(x, 0).$$

This determines a compactification $\overline{X}_d$ of (X, d), which we call the *metric compactification*. The set $\partial_d X = \overline{X}_d \setminus X$ can be naturally thought of as the boundary at infinity of the compactification, so we will call it the *metric boundary* of X.

This boundary can also be understood in a more geometric way by considering certain types of rays which diverge to infinity.

Definition 2.1. *Let (X, d) be a metric space, and T an unbounded subset of $\mathbb{R}^+$ containing 0, and let $\gamma : T \to X$. We say that*

(1) *γ is a* geodesic ray *if*

$$d(\gamma(s), \gamma(t)) = |s - t|$$

for all s, $t \in T$.

(2) *γ is an* almost-geodesic ray *if for every $\varepsilon > 0$, there is an integer N such that*

$$|d(\gamma(t), \gamma(s)) + d(\gamma(s), \gamma(0)) - t| < \varepsilon$$

for all t, $s \in T$ with $t \geq s \geq N$.

(3) *γ is a* weakly-geodesic ray *if for every $y \in X$ and every $\varepsilon > 0$, there is an integer N such that*

$$|d(\gamma(t), \gamma(0)) - t| < \varepsilon$$

and

$$|d(\gamma(t), y) - d(\gamma(s), y) - (t - s)| < \varepsilon$$

for all t, $s \in T$ with t, $s \geq N$.

Every geodesic ray is an almost-geodesic ray and Rieffel showed that every almost-geodesic ray is a weakly-geodesic ray. The significance of weakly-geodesic rays is that their limits are the points of the metric compactification for a large class of metric spaces. Recall that a metric or metric space is *proper* if every closed ball of finite radius is compact.

Theorem 2.1 (Rieffel)**.** *Let (X, d) be a complete, locally compact metric space, and let $\gamma : T \to X$ be a weakly geodesic ray in X. Then*

$$\lim_{t \to \infty} f(\gamma(t))$$

exists for every $f \in \mathcal{G}(X,d)$, and defines an element of $\partial_d X$. Conversely, if d is proper and if (X,d) has a countable base, then every point of $\partial_d X$ is determined as above by a weakly-geodesic ray.

For our purposes, there is one part of the proof of this theorem which is of interest, since it tells us how to construct weakly-geodesic rays from arbitrary sequences which converge to points on the metric boundary:

Lemma 2.2. *Let (X,d) be a complete, proper metric space with a countable base, and let x_n be a sequence which converges to a point $\omega \in \partial_d X$. Then if we choose a subsequence x_{n_k} such that $x_{n_0} = x_0$ and $d(x_{n_k}, x_0) > d(x_{n_l}, x_0)$ for $k > l$, then the function $\gamma : T \to X$ where $T = \{d(x_{n_k}, x_0) : k = 0, 1, \ldots\}$ and $\gamma(t) = x_{n_k}$ for $t = d(x_{n_k}, x_0)$ is a weakly geodesic ray.*

The proof of this lemma is contained as part of [3, Theorem 4.7], and is explicitly pulled out in the discussion of this theorem in [4].

Note also that this weakly-geodesic ray satisfies a slightly stronger condition than a general weakly-geodesic ray, since $d(\gamma(t), \gamma(0)) = t$ instead of approximating it.

The question raised by Rieffel is when can we use almost-geodesic rays in the place of weakly-geodesic rays to find all points on th emetric boundary. To this end he defined a point $\omega \in \partial_d X$ to be a *Busemann point* if there is some almost-geodesic ray γ such that

$$\lim_{t \to \infty} \gamma(t) = \omega.$$

If there is no such almost-geodesic ray, the boundary point is a *non-Busemann point*. This Rieffel's problem can be succinctly stated as:

Question 1. *Given a metric space (X,d), is every point of $\partial_d X$ a Busemann point?*

We now are in a position to prove the main theorems of this paper, which are generalizations of Theorem 2.2 and Theorem 2.3 of [4], and answer the question.

Theorem 2.3. *Let (X,d) be a proper, separable metric space. Assume that given any finite subset F of X, and any $\varepsilon > 0$, there is some number $M_{F,\varepsilon}$ such that if $c \in X$ with $d(F,c) > M_{F,\varepsilon}$, then there is a point $w \in X$ with $d(F,w) \leq M_{F,\varepsilon}$ such that*

$$d(x,w) + d(w,c) < d(x,c) + \varepsilon$$

for every $x \in F$. Then every point on the metric boundary is a Busemann point.

Proof. Let $\omega \in \partial_d X$, and let $\gamma : T \to X$ be a weakly geodesic ray which converges to ω. Without any loss of generality (using the construction from Lemma 2.2) we can assume that T is countable and $d(\gamma(t), \gamma(0)) = t$ for all $t \in T$. We seek an almost-geodesic ray which converges to ω.

Since X is separable it has a countable, dense subset, and let $\{x_k : k \in \mathbb{N}\}$ be an enumeration of this set. Without loss of generality, we may assume that $x_0 = \gamma(0)$.

For each n we will inductively find numbers m_n, vertices w_{m_n}, and unbounded subset T_n of T with the following properties:

(1) $d(w_{m_n}, \gamma(0)) = m_n$.
(2) for all $t \in T_n$, and all $k \leq n$,

$$d(x_k, w_{m_n}) + d(w_{m_n}, \gamma(t)) < d(x_k, \gamma(t)) + 2^{-n}.$$

(3) if $n \geq 1$ for all $t \in T_n$, and all $k < n$

$$d(w_{m_k}, w_{m_n}) + d(w_{m_n}, \gamma(t)) < d(w_{m_k}, \gamma(t)) + 2^{-n}.$$

Let $m_0 = 0$, $w_0 = \gamma(0)$, and $T_0 = T$. All conditions are trivially satisfied.

Given m_n, w_{m_n} and T_n, then let $F = \{x_0, x_1, \dots, x_n, w_{m_1}, \dots, w_{m_n}\}$, so we can find a number $M_{F,2^{-n-2}}$ from the hypotheses.

For each $t \in T_n$ with $d(F, \gamma(t)) > M_{F,2^{-n-2}}$, we can find a point z_t such that

$$d(x, z_t) + d(z_t, \gamma(t)) < d(x, \gamma(t)) + 2^{-n-2}$$

for all $x \in F$ and $d(F, z_t) \leq M_{F,2^{-n-2}}$. Now $B(F, M_{F,2^{-n-2}}) = \{y : d(F, y) \leq M_{F,2^{-n-2}}\}$ is compact since (X, d) is proper and F is finite, so it is totally bounded. Hence we can find a finite collection of sets B_k which cover $B(F, M_{F,2^{-n-2}})$ and which have diameter no greater than 2^{-n-3}. Hence one of these sets, say B_k, must contain an infinite number of the points z_t. Let $T_{n+1} = \{t \in T_n : z_t \in B_k\}$, and choose $m_{n+1} = d(\gamma(0), z_{t_0})$ and $w_{m_{n+1}} = z_{t_0}$ for some $t_0 \in T_{n+1}$. Then for any $t \in T_{n+1}$ we have

$$\begin{aligned} d(x, w_{m_{n+1}}) + d(w_{m_{n+1}}, \gamma(t)) &\leq d(x, z_t) + d(z_t, \gamma(t)) + 2(2^{-n-3}) \\ &\leq d(x, \gamma(t)) + 2^{-n-1}, \end{aligned}$$

for all $x \in F$.

So w_{m_n}, m_n and T_n satisfy all three conditions of the claim.

Now let $\gamma' : T' \to X$, where $T' = \{m_n : n \in \mathbb{N}\}$ and $\gamma'(m_n) = w_{m_n}$. We claim that this is an almost-geodesic ray. By construction, given any $\varepsilon > 0$, there is some N such that if $t \geq s \geq N$, with $\gamma'(s) = w_{m_n}$ and $\gamma'(t) = w_{m_k}$, then for some $u \in T$ we have

$$\begin{aligned} d(x_0, \gamma'(s)) &< d(x_0, \gamma(u)) - d(\gamma'(s), \gamma(u)) + \varepsilon/2, \qquad \text{and} \\ d(\gamma'(s), \gamma'(t)) &< d(\gamma'(s), \gamma(u)) - d(\gamma'(t), \gamma(u)) + \varepsilon/2. \end{aligned}$$

Then, since $d(\gamma(0), \gamma'(t)) = t$ we have

$$\begin{aligned} t = d(\gamma(0), \gamma'(t)) &\leq d(x_0, \gamma'(s)) + d(\gamma'(s), \gamma'(t)) \\ &< d(x_0, \gamma(u)) - d(\gamma'(t), \gamma(u)) + \varepsilon \\ &\leq d(\gamma(0), \gamma'(t)) + \varepsilon = t + \varepsilon, \end{aligned}$$

so

$$|d(\gamma(0), \gamma'(s)) + d(\gamma'(s), \gamma'(t)) - t| < \varepsilon.$$

The same argument, with x_k in the place of $\gamma(0)$ allows us to conclude that there is some N such that for every $t \geq s \geq N$,

$$d(x_k, \gamma'(s)) + d(\gamma'(s), \gamma'(t)) < d(x_k, \gamma'(t)) + \varepsilon.$$

Finally, we claim that this almost-geodesic ray converges to ω. For any x_k and any $\varepsilon > 0$, we can find an n so that $n > k$ and $2^{-n} < \varepsilon$. Then we have for s, $t \in T'$

with s, $t \geq m_n$,

$$\begin{aligned}
|\varphi_{x_k}(\gamma(t)) - \varphi_{x_k}(\gamma'(s))| \\
&= |d(\gamma(t), x_0) - d(\gamma(t), x_k) - d(\gamma'(s), x_0) + d(\gamma'(s), x_k)| \\
&\leq |d(\gamma(t), w_{m_n}) + d(w_{m_n}, x_0) - d(\gamma(t), w_{m_n}) - d(w_{m_n}, x_k) - \\
&\qquad d(\gamma'(s), w_{m_n}) - d(w_{m_n}, x_0) + d(\gamma'(s), w_{m_n}) + d(w_{m_n}, x_k)| + \varepsilon \\
&< \varepsilon.
\end{aligned}$$

Hence

$$\lim_{t \in T'} \varphi_{x_k}(\gamma'(t)) = \lim_{t \in T'} \varphi_{x_k}(\gamma(t)) = \varphi_{x_k}(\omega).$$

For general x, given any $\varepsilon > 0$, we can find k such that $d(x_k, x) < \varepsilon/4$, and then for t and s large enough,

$$\begin{aligned}
|\varphi_x(\gamma(t)) - \varphi_x(\gamma'(s))| &< |\varphi_{x_k}(\gamma(t)) - \varphi_{x_k}(\gamma'(s))| + \varepsilon/2 \\
&< |\varphi_{x_k}(\omega) - \varphi_{x_k}(\omega)| + \varepsilon = \varepsilon.
\end{aligned}$$

So

$$\lim_{t \in T'} \varphi_x(\gamma'(t)) = \lim_{t \in T'} \varphi_x(\gamma(t)) = \varphi_x(\omega)$$

for all $x \in X$, and so

$$\omega = \lim_{t \in T'} \gamma'(t).$$

Hence ω is a Busemann point. □

Theorem 2.4. *Let (X, d) be a proper metric space with a countable base. If there is some finite set F and $\varepsilon > 0$, such that for every $n \in \mathbb{N}$, there is a point c_n such that $d(F, c_n) > n$ and for all x with $d(F, x) \leq n$ we have*

$$d(y, x) + d(x, c_n) \geq d(y, c_n) + \varepsilon,$$

for all $y \in F$, then there is a point in $\partial_d X$ which is not a Busemann point.

Proof. Let x_0 is the base point for the functions φ_x, and let $c_0 = x_0$.

The points c_n are a sequence in the compact set $\bar{X}_d$, so there is a convergent subsequence c_{n_k}. Since $d(F, c_n)$ are unbounded, the points c_{n_k} must head to infinity, and hence the limit point of c_{n_k} is an element ω of $\partial_d X$. From Lemma 2.2, we have a weakly-geodesic ray $\gamma : T \to X$, corresponding to a subsequence of c_{n_k}, which converges to that point on the boundary and which also satisfies the condition $d(\gamma(t), x_0) = t$.

Now assume that ω is a Busemann point, so we can find an almost-geodesic ray $\gamma' : T' \to X$ with $\gamma'(0) = x_0$ which converges to ω. For all $x \in X$, we must therefore have

$$\lim_{t \to \infty} \varphi_x(\gamma(t)) = \lim_{t \to \infty} \varphi_x(\gamma'(t)) = \varphi_x(\omega).$$

So for every x and every $\delta > 0$, we can find some $N_{x,\delta}$, such that $|\varphi_x(\gamma'(t)) - \varphi_x(\omega)| < \delta/2$ and $|\varphi_x(\gamma(s)) - \varphi_x(\omega)| < \delta/2$, for all $s \in T$, $t \in T$, with t, $s \geq N_{x,\delta}$. Also, since γ' is almost-geodesic, and hence weakly-geodesic, we may find M_δ such that

$$\begin{aligned}
|d(\gamma'(t), \gamma'(s)) - (t - s)| &< \delta \\
|d(\gamma'(t), x_0) - t| &< \delta
\end{aligned}$$

for all $t \geq s \geq M_\delta$, with s, $t \in T'$.

In particular, for $y \in F$, we have

$$d(\gamma'(t), y) = d(\gamma'(t), x_0) - \varphi_y(\gamma'(t)) < t - \varphi_y(\omega) + \varepsilon/4$$

for all $t \in T'$ with $t \geq \max\{N_{y,\varepsilon/4}, M_{\varepsilon/8}\}$. But also

$$d(\gamma(t), y) = d(\gamma(t), x_0) - \varphi_y(\gamma(t)) > t - \varphi_y(\omega) - \varepsilon/4$$

for all $t \in T$ with $t \geq N_{y,\varepsilon/4}$. Let $N = \max\{N_{y,\varepsilon/4} : y \in F\} \cup \{M_{\varepsilon/8}\}$.

Now, fix a particular $s \geq N$, and consider $\gamma'(s)$. Then we have

$$\begin{aligned}\varphi_{\gamma'(s)}(\gamma(t)) &> \varphi_{\gamma'(s)}(\gamma'(t')) - \varepsilon/4 \\ &> d(\gamma'(t'), x_0) - d(\gamma'(t'), \gamma'(s)) - \varepsilon/4 \\ &> t' - (t' - s) - \varepsilon/2 = s - \varepsilon/2\end{aligned}$$

for some $t' \geq N_{\gamma'(s),\varepsilon/4}$ and all $t \in T$ with $t \geq s$ and $t \geq N_{\gamma'(s),\varepsilon/4}$. But this means that for such t,

$$d(\gamma(t), \gamma'(s)) = d(\gamma(t), x_0) - \varphi_{\gamma'(s)}(\gamma(t)) < t - s + \varepsilon/2,$$

and so for all $y \in F$,

$$\begin{aligned}d(\gamma(t), \gamma'(s)) + d(\gamma'(s), y) &= (t - s + \varepsilon/2) + (s - \varphi_y(\omega) + \varepsilon/4) \\ &= t - \varphi_y(\omega) + 3\varepsilon/4 \\ &< d(\gamma(t), y) + \varepsilon.\end{aligned}$$

In particular, for t large enough that $\gamma(t) = c_n$ with $n \geq d(F, \gamma'(s))$, this tells us that

$$d(y, \gamma'(s)) + d(\gamma'(s), c_n) < d(y, c_n) + \varepsilon,$$

for all $y \in F$, which contradicts our assumption, and so ω cannot be a Busemann point. □

It is perhaps worthwhile noting that these conditions can be re-cast in terms of closed balls of finite radius.

Corollary 2.5. *Let (X, d) be a proper, separable metric space. Assume that given any closed ball $B(y, r)$ in X, and any $\varepsilon > 0$, there is some number $M_{y,r,\varepsilon}$ such that if $c \in X$ with $d(B(y, r), c) > M_{y,r,\varepsilon}$, then there is a point $w \in X$ with $d(B(y, r), w) \leq M_{y,r,\varepsilon}$ such that*

$$d(x, w) + d(w, c) < d(x, c) + \varepsilon$$

for every $x \in B(y, r)$. Then every point on the metric boundary is a Busemann point.

Proof. Given any finite subset F of (X, d), there is some ball of finite radius $B(y, r)$ such that $F \subseteq B(y, r)$. Then if you let $M_{F,\varepsilon} = M_{y,r,\varepsilon}$ then if $c \in X$ with $d(B(y, r), c) > M_{y,r,\varepsilon}$, then there is a point $w \in X$ with $d(B(y, r), w) \leq M_{y,r,\varepsilon}$ such that

$$d(x, w) + d(w, c) < d(x, c) + \varepsilon$$

for every $x \in B(y, r)$, and hence for every $x \in F$. So by Theorem 2.3, every point on the metric boundary is a Busemann point. □

Corollary 2.6. *Let (X, d) be a proper metric space with a countable base. If there is some closed ball of finite radius $B(y, r)$ and some $\varepsilon > 0$, such that for every $n \in \mathbb{N}$, there is a point c_n such that $d(B(y, r), c) > n$ and for all x with $d(B(y, r), x) \leq n$*

$$d(z, x) + d(x, c) \geq d(z, c) + \varepsilon,$$

for all $z \in B(y, r)$ then there is a point in $\partial_d X$ which is not a Busemann point.

Proof. Given such a ball $B(y, r)$ and such an ε, we know that since (X, d) is proper, $B(y, r)$ is compact and hence totally bounded. As a result there is a finite subset F of $B(y, r)$ such that the sets $\{B(x, \varepsilon) : x \in F\}$ cover $B(y, r)$. But then given any $n \in \mathbb{N}$ if x is any point with $d(F, x) \leq n$, we have $d(B(y, r), x) < d(F, x)$, since $F \subseteq B(y, r)$, and hence

$$d(z, x) + d(z, c_n) \geq d(y, c_n) + \varepsilon.$$

for all $z \in B(y, r)$, and hence for all $z \in F$. So we have shown that the condition of Theorem 2.4 apply, and so there is a point in $\partial_d X$ which is not a Busemann point. □

3. Metric Groups

In [4] we were able to use the symmetries of Cayley graphs to provide a simpler characterization of the existence of non-Busemann points in terms of triangles within the Cayley graph. Since the principal difference between the theorems presented in this paper and the previous work is that a pair of points is replaced by a finite set of points, there does not appear to be an elegant analogue of these theorems. However, we can find a theorem which is similar in nature, if much more limited, and which may be useful.

Definition 3.1. *Let (X, d) be a metric space and (a, b, c) be a triple of points. If for all $\varepsilon > 0$ and for all $x \in X$, we have*

$$d(a, x) + d(x, c) > d(a, c) + \varepsilon$$
$$d(b, x) + d(x, c) > d(b, c) + \varepsilon$$
$$d(a, x) + d(x, b) > d(a, b) + \varepsilon$$

*then the triple (a, b, c) is said to be an ε-*rigid triple*. We define the* perimeter *of such a triple to be*

$$P(a, b, c) = d(a, b) + d(b, c) + d(c, a).$$

With this definition, we have the following result which is analagous to [4, Proposition 3.3].

Theorem 3.1. *Let G be a group and d a left- (or right-) invariant metric such that (G, d) is a proper, separable metric space. If there exists $M \in \mathbb{R}$ such that for all $m \in \mathbb{N}$, there is an ε-rigid triple (a_m, b_m, c_m) with $d(a_m, b_m) \leq w$ and $P(a_m, b_m, c_m) \geq m$, then there is a point on the metric boundary which is not a Busemann point.*

Proof. Since the metric is left invariant, the ε-rigid triple (a, b, c) yields another ε-rigid triple (e, y_m, z_m) by left-multiplication by a_{-1}, and these ε-rigid triple also satisfy the hypotheses of the theorem.

We will construct a finite set $F = \{e, y\}$ that satisfies the conditions of Theorem 2.4 to prove the existence of a non-Busemann point. For all m, y_m is contained in the closed ball $B(e, M)$, because $d(e, y_m) = d(a_m, b_m) \leq M$. Since (X, d) is proper, $B(e, M)$ is compact, and the sets $\{N(x, \varepsilon/4) : xinC\}$ cover C, and so there is a finite subcover. Since there are y_m, one of these neighbourhoods, say $N(y, \varepsilon/4)$, contains infinitely many y_m.

We now show that the set $F = \{e, y\}$ satisfies the conditions of Theorem 2.4. Let $n \in N$. Then there is a rigid triple (e, y_m, z_m) with $y_m \in N(y, \varepsilon/4)$ and perimeter big enough so that $d(e, z_m) > n$ and $d(y_m, z_m) > n$, since for each m, $d(e, y_m)$ is bounded by M, but the perimeter gets arbitrarily large. Now define $c_n = z_m$, so that $d(F, c_n) > n$, and consider any x such that $d(F, x) > n$. Because (e, y_m, z_m) is an ε-rigid triple, we have that for any x

$$d(y_m, x) + d(x, z_m) > d(y_m, z_m) + \varepsilon.$$

But $y_m \in N(y, \varepsilon/4)$, so

$$\begin{aligned} d(y, x) + d(x, z_m) &\geq d(y_m, x) - \varepsilon/4 + d(x, z_m) \\ &> (d(y_m, z_m) + \varepsilon) - \varepsilon/4 \\ &> (d(y, z_m) - \varepsilon/4) + \varepsilon - \varepsilon/4 \\ &> d(y, z_m) + \varepsilon/2 \\ &> d(y, c_n) + \varepsilon/2. \end{aligned}$$

Hence Theorem 2.4 applies with $\{e, y\}$ and $\varepsilon/2$, and there is a non-Busemann point on the metric boundary. □

We do not expect this to be a complete characterisation of those metric groups which have non-Busemann points, and in particular the converse is almost certainly false. However this may prove to be a useful test which can be applied in some situations to prove the existence of non-Busemann points.

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