MAT 708 - Exercises

1. In the notes we used the fact that if μ and ν are positive measures on $(X, \mathcal{M})$ then so is the set function $\rho = \mu + \nu$ defined in the obvious way by $\rho(E) = \mu(E) + \nu(E)$.

   Verify that ρ is in fact a measure.

2. Generalizing the previous problem: let μ and ν be either signed measures or complex measures on a measurable space $(X, \mathcal{M})$. Define $\mu + \nu$ as above, and define $\rho = \lambda\mu$ by $\rho(E) = \lambda\mu(E)$, where $\lambda \in \mathbb{R}$ (if μ is signed) or $\mathbb{C}$ (if μ is complex).

   Show that the finite signed measures form a vector space over $\mathbb{R}$ and that the complex measures form a vector space over $\mathbb{C}$.

   Explain why the set of signed measures is not a vector space with these operations.

3. Show that if μ is a signed measure on X, then $\mu(X) \neq \pm\infty$ if and only if μ is a finite signed measure.

4. Show that if ν is a signed measure on $(X, \mathcal{M})$ and μ_1 and μ_2 are positive measures on the same space such that $\nu = \nu_1 - \nu_2$, then $\nu_1 \geq \nu_+$ and $\nu_2 \geq \nu_-$.

5. Let ν be a signed measure on $(X, \mathcal{M})$. Show that if A_j is a sequence of sets in $\mathcal{M}$ with $A_j \subseteq A_k$ if $j \leq k$, then

   $$\nu(\bigcup_j A_j) = \lim_{j \to \infty} \nu(A_j).$$

   (Try to prove this without using the Jordan Decomposition).

6. Show that if $\nu \ll \mu$ and $\nu \perp \mu$, then $\nu(E) = 0$ for every measurable set E.

7. Suppose μ and ν are finite measures on $(X, \mathcal{M})$ with $\nu \ll \mu$ and let $\lambda = \mu + \nu$. If $f = d\nu/d\lambda$, then show $0 \leq f < 1$ μ-a.e., and $d\nu/d\mu = f/(1-f)$.

8. Show that the uniform norm

   $$\|f\|_u = \sup_{x \in X} |f(x)|$$

   is a norm on vector spaces of $\mathbb{R}$- or $\mathbb{C}$- valued functions.

9. This problem generalizes the concepts introduced in Example 4.1.6 and 4.1.8. You will also need to review the material on Hölder's inequality. This is a hard problem and counts as credit for 3 regular problems if completely solved.

   Recall that if A is a self-adjoint matrix, one can find a unitary matrix which diagonalises it, $A = U^*DU$, where D is a diagonal matrix whose entries are the eigenvalues λ_k of A. In this case, if $f : \mathbb{C} \to \mathbb{C}$ is a function, we define $f(A) = U^* f(D) U$, where $f(D)$ is the diagonal matrix with entries $f(\lambda_k)$ on the diagonal.

   (a) Show that A^*A is self-adjoint for any $A \in M_n$. Define $|A| = (A^*A)^{1/2}$, and show that $|A|$ is self-adjoint.

   (b) For $1 \leq p < \infty$, we define the Schatten p-norms on M_n by

   $$\|A\|_p = \operatorname{tr}(|A|^p)^{1/p}.$$

   Show that

   $$\|A\|_p = \left(\sum_{k=1}^{n} |\lambda_k|^p \right)^{1/p} = \|(\lambda_1, \ldots, \lambda_n)\|_p,$$

   where λ_k are the eigenvalues of $|A|$, and conclude that $\|kA\|_p = |k|\|A\|_p$ and $\|A\|_p = 0$ iff $A = 0$.

   (c) Let

   $$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}.$$

   Find $\|A\|_1$, $\|A\|_2$ and $\|A\|_{10}$.

   (d) Use Hölder's inequality for ℓ^p-spaces to prove Hölder's inequality for Schatten p-norms: if p and q are conjugate exponents, then

   $$\|AB\|_1 \leq \|A\|_p \|B\|_q$$

   for all A and $B \in M_n$.

(e) Use Hölder's inequality to show that the triangle inequality holds for the Schatten p-norms for $1 < p < \infty$. Conclude that the Schatten p-norms are in fact norms for $1 < p < \infty$.

(f) Show that the Schatten 1-norm is in fact a norm.

The point of this problem is part (e) and (f), so if you can show that the Schatten p-norms are in fact norms for $1 \leq p < \infty$ using other means, you will get full credit for this problem.

10. Let H be a finite dimensional Hilbert space, and let $T \in B(H)$. Show that the operator norm of T is given by

$$\|T\| = \max\{\lambda^{1/2} : \lambda \text{is an eigenvalue of } T^*T\}.$$

Briefly discuss why, in light of this and the previous problem, it is reasonable to call the operator norm of a matrix regarded as a linear transformation the "Schatten ∞-norm."

11. A subset K of a vector space V is *convex* if

$$\sum_{k=1}^{n} \lambda_k v_k \in K,$$

for all $v_k \in K$ and $\lambda_k \in [0,1]$ such that

$$\sum_{k=1}^{n} \lambda_k = 1.$$

A subset K of a vector space is *circled* if $\lambda v \in K$ for every every $v \in K$ and every scalar λ with $|\lambda| = 1$.

Show that the unit ball of a normed vector space is a convex, circled set.

12. Given a convex, circled set $K \subseteq V$ containing 0, define the *gauge* of K to be the function $\rho_K : V \to [0,\infty]$ defined by

$$\rho_K(v) = \inf\{\lambda \geq 0 : v \in \lambda K\}.$$

Show that if ρ_K is finite (ie. K does not contain any rays), then ρ_K is a seminorm.

13. Let $(X,\mathcal{M})$ be a measurable space, and let M be the vector space of all finite signed measures on $(X,\mathcal{M})$ (see problem 2). Define

$$\|\mu\| = |\mu|(X),$$

and show that this is a norm on M. Show that M is complete, and hence a Banach space.

14. Let H be an infinite dimensional Hilbert space. Show that the closed unit ball of H is not compact.

15. Let $(X,\mathcal{M},\mu)$ be a finite measure space, and let $1 \leq p < q \leq \infty$. We know that since $L^q(X) \subseteq L^p(X)$, the identity map $Tf = f$ is a linear map from $L^q(X)$ to $L^p(X)$. Show that for $q < \infty$,

$$\|T\| = \mu(X)^{q/p(p-q)}.$$

Show that if $q = \infty$, then

$$\|T\| = \mu(X)^{1/p}.$$

Show that if $1 \leq p < q \leq \infty$, the identity map $T : \ell^p(X) \to \ell^q(X)$ has operator norm $\|T\| = 1$.

16. Let c_∞ be the set of all sequences of complex numbers $x = (x_n)_{n=1}^{\infty}$, such that

$$\lim_{n\to\infty} x_n = 0.$$

Show that this is a Banach space with the uniform norm

$$\|x\|_u = \sup_{n\in\mathbb{N}} |x_n|.$$

Show that $c_\infty^* = \ell^1(\mathbb{N})$.

17. Find an example of an orthonormal basis of the Hilbert space $H = L^2(\mathbb{R},m)$.

18. Let $1 \leq p < q \leq \infty$. Give an example of a function which is in $L^r(\mathbb{R},m)$ for all r with $p < r < q$, but which is not in $L^p(\mathbb{R},m)$ or $L^q(\mathbb{R},m)$.

19. Let $(X,\mathcal{M})$ be a measurable space, with (positive) measures μ and ν, and $\nu \ll \mu$. If $1 \leq p \leq \infty$, show that a function $f \in L^p(X,\nu)$ is an element of $L^p(X,\mu)$ if

$$\frac{d\nu}{d\mu} \in L^\infty(X,\mu).$$

20. Let H and K be Hilbert spaces. Let $H \oplus K$ be the set $H \times K$ with the standard vector space operations, and inner product

$$\langle (x, v), (y, u) \rangle = \langle x, y \rangle + \langle u, v \rangle$$

for all x, $y \in H$ and u, $v \in K$. Show that $H \oplus K$ is a Hilbert space.

Show that $(x_n, v_n) \to (x, v)$ in $H \oplus K$ if and only if $x_n \to x$ in H and $v_n \to v$ in K.

Show that if $\{e_\alpha : \alpha \in I\}$ is an orthonormal basis for H and $\{f_\beta : \beta \in J\}$ is an orthonormal basis for K, then $\{(e_\alpha, 0) : \alpha \in I\} \cup \{(0, f_\beta) : \beta \in J\}$ is an orthonormal basis for $H \oplus K$.

21. Let $H = L^2([0,1])$ and $D : C^\infty([0,1]) \to C^\infty([0,1])$ be the differentiation operator $Df = f'$. Show that the graph of D,

$$\mathcal{G}_D = \{(f, Df) : f \in C^\infty([0,1])\}$$

is a closed subspace of $H \oplus H$, ie. if

$$(f_n, Df_n) \to (f, g)$$

then $(f, g) \in \mathcal{G}_D$, ie. $g = Df$.

22. Let H and K be Hilbert spaces. Let $H \odot K$ be the vector space spanned by vectors $u \otimes v$ for $u \in H$ and $v \in K$, where $\lambda(u \otimes v) = (\lambda u) \otimes v = u \otimes (\lambda v)$ and $u_1 \otimes v + u_2 \otimes v = (u_1 + u_2) \otimes v$. In other words, a typical vector looks like

$$\sum_{k=1}^{n} \lambda_k u_k \otimes v_k.$$

Show that

$$\left\langle \sum_{k=1}^{n} \lambda_k u_k \otimes v_k, \sum_{j=1}^{m} \mu_j w_j \otimes z_j \right\rangle = \sum_{k=1}^{n} \sum_{j=1}^{m} \langle u_k, w_j \rangle \langle v_k, z_j \rangle$$

defines an inner product on $H \odot K$.

We define the *tensor product* $H \otimes K$ be the completion of this vector space in the norm derived from the inner product.

Show that if $\{e_\alpha : \alpha \in I\}$ is an orthonormal basis for H and $\{f_\beta : \beta \in J\}$ is an orthonormal basis for K, then $\{e_\alpha \otimes f_\beta : \alpha \in I, \beta \in J\}$ is an orthonormal basis for $H \otimes K$.