Quiz Solutions

These are suggested solutions. Other solutions may be possible.

Quiz 1

1. Define what it means for an integer to be *square*. For example, the integers 0, 1, 4, 9, and 16 are square. Your definition should begin:

> An integer x is called *square* provided...

Answer:
An integer x is called square *provided there is an integer n such that $n^2 = x$.*

2. Rewrite each of the following statements in the form "If A then B."

a. "The product of an odd integer and an even integer is even."

b. "The square of an odd integer is odd."

c. "The square of a prime number is not prime."

d. "The product of two negative numbers is negative."

Answer:

a. "If a is an odd integer and b is an even integer, then ab is even."

b. "If a is an odd integer then a^2 is odd."

c. "If p is a prime number then p^2 is not prime."

d. "If a and b are negative numbers then ab is negative."

Quiz 2

1. Prove that an integer is odd if and only if it is the sum of two consecutive integers.

Answer:
($\Rightarrow$) We will show that if an integer a is odd, then it is the sum of two consecutive integers. Let a be an odd integer. By Definition 1.3, there is some integer n such that $a = 2n + 1$. But

$$a = 2n + 1 = n + (n + 1)$$

and n and $n + 1$ are consecutive integers. Therefore a is the sum of two consecutive integers.
($\Leftarrow$) We will show that if an integer a is the sum of two consecutive integers n and $n + 1$, then it is odd. Let a be the sum of the consecutive integers n and $n + 1$. Then

$$a = n + (n + 1) = 2n + 1.$$

So there is an integer n such that $a = 2n + 1$, so a is odd by definition 1.3.

2. Disprove: Two right-angled triangles have the same area if and only if the lengths of their hypotenuses are the same.

Answer: (Note: there are many different ways of doing this. A diagram would help, but it is hard for me to produce one on the computer.)

Let ABC be the right angle triangle with angle ABC a right angle, the length of leg AB equal to 4 and the length of leg BC equal to 3. The area of this triangle is 6 units and Pythagoras' Theorem says that such a triangle has hypotenuse length $\sqrt{4^2 + 3^2} = 5$.

Let DEF be the right angle triangle with angle DEF a right angle, the length of leg DE equal to 6 and the length of leg EF equal to 2. The area of this triangle is 6 units and Pythagoras' Theorem says that such a triangle has hypotenuse length $\sqrt{6^2 + 1^2} = \sqrt{37}$.

Therefore ABC and DEF are two triangles with the same area, but hypotenuses of different lengths, so the statement is not true.

Quiz 3

1. Let $A = \{1, 2, 3, 4\}$, $B = \{3, 4, 5, 6\}$ and the universe of objects that we are considering be $\{1, 2, \ldots, 10\}$. Find:

a. $A \cap B$.

b. $A \cup B$.

c. $A \setminus B$.

d. $B \setminus A$.

e. $A \triangle B$.

f. A^c.

g. B^c.

h. $\mathcal{P}(A)$.

Answer:

a. $A \cap B = \{3, 4\}$.

b. $A \cup B = \{1, 2, 3, 4, 5, 6\}$.

c. $A \setminus B = \{1, 2\}$.

d. $B \setminus A = \{5, 6\}$.

e. $A \triangle B = \{1, 2, 5, 6\}$.

f. $A^c = \{5, 6, 7, 8, 9, 10\}$.

g. $B^c = \{1, 2, 7, 8, 9, 10\}$.

h. $\mathcal{P}(A) = \{\emptyset,$
$\{1\}, \{2\}, \{3\}, \{4\},$
$\{1, 2\}, \{1, 3\}, \{1, 4\}, \{2, 3\}, \{2, 4\}, \{3, 4\},$
$\{1, 2, 3\}, \{1, 2, 4\}, \{1, 3, 4\}, \{2, 3, 4\},$
$\{1, 2, 3, 4\}\}$.

2. Write the statement "for every integer x there exists a prime number p such that p divides x" symbolically (use P for the set of prime numbers). Write down its negation symbolically. Write the negation in English. Which statement is true?

Answer:
Symbolically, we have $\forall\ x \in \mathbb{Z}, \exists\ p \in P, p \mid x$.
The negation is

$$\begin{aligned}\neg(\forall\ x \in \mathbb{Z}, \exists\ p \in P, p \mid x) &= \exists x \in \mathbb{Z}, \neg(\exists p \in P, p \mid x)\\ &= \exists x \in \mathbb{Z}, \forall p \in P, \neg(p \mid x).\end{aligned}$$

In English, the negation would be "There is an integer x such that for every prime number p, p does not divide x." A more natural way of saying this would be "There is some integer which cannot be divided by any prime number."

The negation is true, because 1 is an integer which cannot be divided by any prime number.

Quiz 4

1. Prove that if the sum of two primes is prime, then one of the primes is 2.

Answer:
Let p and q be two prime numbers whose sum, $p + q$*, is also prime.*

Assume, for sake of contradiction, that both $p \neq 2$ *and* $q \neq 2$*. So in particular,* $p > 2$ *and* $q > 2$*. Then both p and q are not even, since* 2 *cannot divide a prime number greater than* 2*. Therefore both p and q are odd by a proposition proved in class. So there are integers n and m such that* $p = 2n + 1$ *and* $q = 2m + 1$*. Therefore* $p + q = (2n + 1) + (2m + 1) = 2(m + n + 1)$ *is divisible by 2. But if p and q are greater than* 2*, so is* $p + q$*. So* $p + q$ *is a prime number greater than* 2 *which is divisible by* 2*.*

This is a contradiction, and p or q must be 2*.*

2. Prove using induction that

$$1 + 3 + 5 + \cdots + (2n - 1) = n^2.$$

Answer:
If $n = 1$*, then* $1 = 1^2$*, so the proposition is true in this case.*
Assume that $1 + 3 + 5 + \cdots + (2n - 1) = n^2$*.*

Now

$$\begin{aligned} 1+3+5+\cdots+(2n-1)+(2(n+1)-1) &= (1+3+5+\cdots+(2n-1))+2n+1 \\ &= n^2+2n+1 \\ &= (n+1)^2. \end{aligned}$$

So by the principle of mathematical induction, $1+3+5+\cdots+(2n-1)=n^2$ *for all natural numbers* n*.*

Quiz 5

1. Prove the distributive rule $(A \cap B) \times C = (A \times C) \cap (B \times C)$.

Answer:

Let $(x,y) \in (A \cap B) \times C$*. Then* $x \in A \cap B$ *and* $y \in C$*. Therefore* $x \in A$ *and* $x \in B$*. Hence* $(x,y) \in A \times C$ *and* $(x,y) \in B \times C$*. Therefore* $(x,y) \in (A \times C) \cap (B \times C)$*.*

Let $(x,y) \in (A \times C) \cap (B \times C)$*. Then* $(x,y) \in A \times C$ *and* $(x,y) \in B \times C$ *and so* $x \in A$ *and* $x \in B$ *and* $y \in C$*. Therefore* $x \in A \cap B$ *and so* $(x,y) \in (A \cap B) \times C$*.*

So $(A \cap B) \times C = (A \times C) \cap (B \times C)$*.*

2. Define a relation $\equiv$ on $\mathbb{Z}$ by $a \equiv b$ if $2 \mid (a-b)$. Show that $\equiv$ is an equivalence relation.

Answer:

We will show that $\equiv$ *is reflexive, symmetric and transitive.*

Let $a \in \mathbb{Z}$*. Then* $a - a = 0$ *and* $2 \mid 0$*, so* $2 \mid (a-a)$*. Hence* $a \equiv a$*. Therefore* $\equiv$ *is reflexive.*

Let $a, b \in \mathbb{Z}$ *such that* $a \equiv b$*. Then* $2 \mid (a-b)$*, so there is some* $n \in \mathbb{Z}$ *such that* $a - b = 2n$*. But then* $b - a = -(a-b) = 2(-n)$*, so* $2 \mid (b-a)$*. Therefore* $b \equiv a$*, and so* $\equiv$ *is symmetric.*

Let $a, b, c \in \mathbb{Z}$ *such that* $a \equiv b$ *and* $b \equiv c$*. Then* $2 \mid (a-b)$ *and* $2 \mid (b-c)$ *so there are integers* n *and* m *such that* $a-b=2n$ *and* $b-c=2m$*. But then*

$$a - c = (a-b)+(b-c) = 2(n+m)$$

and so $2 \mid (a-c)$*. Therefore* $a \equiv c$ *and so* $\equiv$ *is transitive.*

Since $\equiv$ *is reflexive, symmetric and transitive, it is an equivalence relation.*

Quiz 6

1. Let $X = \{1,2,3,4\}$ and let $\mathcal{P} = \{\{1\},\{2,3,4\}\}$. Write out the relation $\overset{\mathcal{P}}{\equiv}$ as a set of ordered pairs.

Answer:

$$\overset{\mathcal{P}}{\equiv} = \{(1,1),(2,2),(2,3),(2,4),(3,2),(3,3),(3,4),(4,2),(4,3),(4,4)\}$$

2. Prove that if $\sim$ is an equivalence relation on X, then for any $a, b \in X$ we have $a \in [b]$ if and only if $b \in [a]$.

Answer:

($\Rightarrow$*) Let* $a \in [b]$*. By definition, then* $b \sim a$*. But* $\sim$ *is symmetric, so* $a \sim b$*. Hence* $b \in [a]$*.*

($\Leftarrow$*) This is exactly the same argument, but with* a *and* b *swapped.*

Quiz 7

1. How many of the following types of poker hands are possible? (Consider a deck of cards as consisting of 4 suits ($\heartsuit$, $\diamondsuit$, $\clubsuit$, $\spadesuit$), each with one of each face (2,3,...,10,J,Q,K,A))

   a. Three of a kind: 3 cards with the same face, and two cards with two other (different) faces. Eg. J$\heartsuit$, J$\clubsuit$, J$\diamondsuit$, 7$\diamondsuit$, 4$\heartsuit$.

   b. Straight: The cards can be arranged in sequence by face; the suits are irrelevant. Eg. 8$\heartsuit$, 9$\heartsuit$, 10$\diamondsuit$, J$\diamondsuit$, Q$\heartsuit$.

Answer:

(a) *There are* 13 *possible faces for the* 3 *cards. The cards have* 3 *of the* 4 *possible suits for a card with that face, so this is* $\binom{4}{3}$ *possibilities. The remaining cards have* 2 *of the remaining* 12 *faces, and can have any suit, giving* $\binom{12}{2} \times 4^2$ *possibilities. So the total number of 3 of a kind are:*

$$13 \times \binom{4}{3} \times \binom{12}{2} \times 4^2 = 54912.$$

(b) *There are* 10 *possiblities for the smallest face of the* 5 *cards (allowing A to be low as well as high,* 9 *if you don't allow this). Once you have chosen this, the face value of all the other cards is determined, so all that can vary is the suit. Each card can have any of the* 4 *suits, so that is* 4^5 *possibilities. So the total number of straights is:*

$$10 \times 4^5 = 10240.$$

2. Prove that

$$\binom{n}{k} = \binom{n}{n-k}.$$

Answer:

Using the formula, we have:

$$\begin{aligned}\binom{n}{n-k} &= \frac{n!}{(n-(n-k))!(n-k)!}\\ &= \frac{n!}{k!(n-k)!}\\ &= \binom{n}{k}\end{aligned}$$

OR

Answer:

Consider the number of ways that a series of n coin tosses can have k heads. This amounts to chosing which subset of k of the n tosses comes up heads, so this gives $\binom{n}{k}$ possibilities.

On the other hand, if there are k heads, there are $n - k$ tails, so this is the same as chosing which subset of $n - k$ of the n tosses come up tails. So this gives $\binom{n}{n-k}$ possibilities.

Therefore the two quantities are the same.

OR

Answer:

We know that $(x + y)^n = (y + x)^n$. Now

$$(x + y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k$$

and

$$(y + x)^n = \sum_{j=0}^{n} \binom{n}{j} y^{n-j} x^j.$$

Since these two polynomials agree, we have that the corresponding corefficients must agree, ie. when $j = n - k$ (so $n - j = k$), we have that

$$\binom{n}{k} = \binom{n}{j} = \binom{n}{n-k}.$$