8. The Pigeonhole Principle and Composition

1. Explain why in a group of 367 people, two of them must share a birthday. How many people do you need to guarantee that 3 of them share a birthday?

2. Given a set of seven distinct positive integers, prove that there is a pair whose sum or difference is a multiple of 10. (You may use the fact that if the ones digit of an integer is 0, it is a multiple of 10.)

3. Let $f = \{(1,2),(2,3),(3,1)\}$ and $g = \{(1,1),(2,4),(3,2)\}$. Find $f \circ g$ and $g \circ f$, or explain why they don't exist.

4. Let $f : A \to B$ and $g : B \to A$ be bijections. Prove that $(f \circ g)^{-1} = g^{-1} \circ f^{-1}$.

5. Let $f : A \to A$ be a bijection. Show that $(f^n)^{-1} = (f^{-1})^n$. (You may use the previous problem, if you like.)

6. Let $f : A \to B$ and $g : B \to C$. Prove the following:

   (a) if f and g are onto, so is $g \circ f$.

   (b) if f and g are one-to-one, so is $g \circ f$.

   (c) if f and g are bijections, so is $g \circ f$.

7. Disprove: Let $f,\ g : A \to A$. Then $(f \circ g)^2 = f^2 \circ g^2$.

8. Disprove: Let $f,\ g : A \to A$. Then $f \circ g = g \circ f$.

9. Show that the identity function $\mathbb{I}_A$ on a set A, and the equality relation $E = \{(a,a) : a \in A\}$ are the same relation.