7. Combinatorics

1. A telephone number in the US and Canada is a 10 digit number of which the first digit cannot be a 0 or a 1. How many possible telephone numbers are there?

2. A simple computer language permits variable names that consist of upper- and lower-case letters, and digits, with the restrictions that a variable name cannot start with a digit, and variable names are at most 8 characters long. How many different variable names are there?

3. How many anagrams (including nonsense words) are there of the following words:

   (a) PURPLE

   (b) NEVADA

   (c) APPALACHIAN

   (d) PASTURE

4. 20 knights of the round table sit down at the round table. How many different arrangements of the knights are there, if you consider just the order around the table, not the position that they are sitting in.

5. Twenty people are at a party. If everyone shakes everyone elses hand exactly once, how many handshakes occurred at the party?

6. Prove that

$$\binom{n}{k} = \binom{n}{n-k}.$$

   Now prove it again using a different proof technique.

7. How many rectangles can be formed from an m-by-n chessboard? (For example a 2-by-2 chessboard has 9 such rectangles.)

8. A standard poker hand consists of 5 cards drawn from a deck of 52 cards. How many different poker hands are there?

9. How many of the following types of poker hands are possible? (Consider a deck of cards as consisting of 4 suits ($\heartsuit$, $\diamondsuit$, $\clubsuit$, $\spadesuit$), each with one of each face (2,3,...,10,J,Q,K,A))

   (a) Four of a kind: four cards with the same face, and one other card. Eg. J$\heartsuit$, J$\clubsuit$, J$\diamondsuit$, J$\spadesuit$, 4$\heartsuit$.

   (b) Full house: The hand has 3 cards of one facem and 2 cards of a second face. Eg. J$\heartsuit$, J$\clubsuit$, J$\diamondsuit$, 4$\diamondsuit$, 4$\heartsuit$.

   (c) Three of a kind: 3 cards with the same face, and two cards with two other (different) faces. Eg. J$\heartsuit$, J$\clubsuit$, J$\diamondsuit$, 7$\diamondsuit$, 4$\heartsuit$.

   (d) Flush: All 5 cards have the same suit.

   (e) Straight: The cards can be arranged in sequence by face; the suits are irrelevant. Eg. 8$\heartsuit$, 9$\heartsuit$, 10$\diamondsuit$, J$\diamondsuit$, Q$\heartsuit$.

   (f) Straight flush: a straight and a flush at the same time.

10. Let X be a set of finite cardinality n, and let

$$\begin{bmatrix} n \\ k \end{bmatrix}$$

   be the number of partitions of X into k parts.

   (a) Calculate $\begin{bmatrix} n \\ 1 \end{bmatrix}$.

   (b) Calculate $\begin{bmatrix} n \\ n \end{bmatrix}$.

   (c) Calculate $\begin{bmatrix} n \\ 2 \end{bmatrix}$.

   (d) Prove that

$$\begin{bmatrix} n \\ k \end{bmatrix} = \begin{bmatrix} n-1 \\ k-1 \end{bmatrix} + k \begin{bmatrix} n-1 \\ k \end{bmatrix}.$$

11. Let B_n be the total number of partitions of a set of size n, and let $B_0 = 1$. Calculate B_1, B_2, B_3 and B_4. Prove that

$$B_{n+1} = \sum_{k=0}^{n} \binom{n}{k} B_k.$$

   Use this to find B_5 and B_6.