4. More Induction, Cartesian Products, Relations

1. The Fibonacci sequence is the sequence of numbers $F_1, F_2, \ldots, F_n, \ldots$ defined by letting $F_1 = 1$ and $F_2 = 1$ and then $F_n = F_{n-1} + F_{n-2}$. In other words, each number is the sum of the previous 2.

   Prove that
   $$F_n = \frac{\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n}{\sqrt{5}}.$$

2. Prove that every natural number can be expressed as the sum of distinct powers of 2. For example, $21 = 2^4 + 2^2 + 2^0$.

3. If $A = \{1, 2, 3\}$, $B = \{2, 4, 6\}$, $C = \{1, 2\}$, then find
   (a) $A \times B$.
   (b) $B \times A$.
   (c) C^3.
   (d) $(A \cup B) \times C$.
   (e) $(A \times C) \cup (B \times C)$.
   (f) $A \times B \times C$.
   (g) $(A \times B) \times C$.
   (h) $A \times (B \times C)$.
   (i) $\emptyset \times A$.

4. Is Cartesian product communtative? Associative?

5. Prove the distributive rules:
   (a) $(A \cup B) \times C = (A \times C) \cup (B \times C)$.
   (b) $(A \cap B) \times C = (A \times C) \cap (B \times C)$.
   (c) $(A \setminus B) \times C = (A \times C) \setminus (B \times C)$.

6. Prove that $A \times B = \emptyset$ if and only if one of A or B is the emptyset.

7. Prove that the statement $\forall (a, b) \in A \times B, P(a, b)$ is logically equivalent to $\forall a \in A, b \in B, P(a, b)$. State and prove the corresponding equivalence for $\exists$.

8. Let X be a set, and let $\mathcal{P}(X)$ be the power set of X. Show that the relation $\subseteq$ on $\mathcal{P}(X)$ (ie. A is related to B if $A \subseteq B$) is a partial order.

9. Show that $a \mid b$ is a partial order on $\mathbb{Z}$.

10. Define a relation $\equiv$ on $\mathbb{Z}$ by $a \equiv b$ if $2 \mid (a - b)$. Show that $\equiv$ is an equivalence relation.

11. Let $\preceq$ be a preorder on a set. Define $a \sim b$ if $a \preceq b$ and $b \preceq a$. Show that $\sim$ is an equivalence relation.