4. Proof by Contrapositive and Contradiction

1. Let X and Y be sets. Prove that if $X \setminus Y = X$ then $X \cap Y = \emptyset$.

2. Prove that if the sum of two primes is prime, then one of the primes is 2.

3. Let X and Y be finite sets. Prove that if $|X \cup Y| = |X| + |Y|$ then $X \cap Y = \emptyset$.

4. Let a and b be integers. Prove that if $a + b$ is odd, then one of a or b is odd and the other is even.

5. Let a and b be natural numbers. Prove that if $a \mid b$ then there is a unique integer n such that $b = an$.

6. Let a be an natural number. Prove that if a is not divisible by 3, then there is an integer n such that either $a = 3n + 1$, or $a = 3n + 2$.

7. Prove that if p is prime with $p > 3$, then either $p + 2$ or $p + 4$ is divisible by 3, but not both.

8. Prove using induction that for all natural numbers,

$$1^2 + 2^2 + 3^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}.$$

9. Prove using induction that

$$1 + 3 + 5 + \cdots + (2n - 1) = n^2.$$

10. Let $A_1,\ A_2, \ldots, A_n$ be pairwise disjoint finite sets, that is $A_j \cap A_k = \emptyset$ if $j \neq k$. Prove that

$$|A_1 \cup A_2 \cup \cdots \cup A_n| = |A_1| + |A_2| + \cdots + |A_n|.$$