3. Sets and Quantifiers

1. Let $X = \{4, 5, 7, \{1, 2, 3\}, \{5, 7\}, \{5\}\}$. Which of the following are true and which are false?

   (a) $\emptyset \in X$.

   (b) $\emptyset \subseteq X$.

   (c) $5 \in X$.

   (d) $5 \subseteq X$.

   (e) $\{5\} \in X$.

   (f) $\{5\} \subseteq X$.

   (g) $4 \in X$.

   (h) $4 \subseteq X$.

   (i) $\{4\} \in X$.

   (j) $\{4\} \subseteq X$.

   (k) $7 \in X$.

   (l) $7 \subseteq X$.

   (m) $\{7\} \in X$.

   (n) $\{7\} \subseteq X$.

   (o) $1 \in X$.

   (p) $1 \subseteq X$.

   (q) $\{1\} \in X$.

   (r) $\{1\} \subseteq X$.

   (s) $\{1, 2, 3\} \in X$.

   (t) $\{1, 2, 3\} \subseteq X$.

   (u) $\{5, 7\} \in X$.

   (v) $\{5, 7\} \subseteq X$.

2. Let $A = \{1, 2, 3, 4\}$, $B = \{3, 4, 5, 6\}$ and the universe of objects that we are considering be $\{1, 2, \ldots, 10\}$. Find:

   (a) $A \cap B$.

   (b) $A \cup B$.

   (c) $A \setminus B$.

   (d) $B \setminus A$.

   (e) $A \triangle B$.

   (f) A^c.

   (g) B^c.

   (h) $\mathcal{P}(A)$.

3. What is $\mathcal{P}(\emptyset)$?

4. Let a and b be integers and let $A = \{x \in \mathbb{Z} : a|x\}$ and $B = \{x \in \mathbb{Z} : b|x\}$. Prove that if $a \mid b$ then $B \subseteq A$.

5. In a class of 30 students, 20 understand set theory and 15 understand boolean algebra. If every student understands something, how many understand both set theory and boolean algebra?

6. Prove Theorem 5.1 (xvii).

7. Show that $|A \triangle B| = |A \setminus B| + |B \setminus A|$.

8. If A, B and C are sets, illustrate using a Venn Diagram the set $A \cup (B \setminus C)$.

9. Explain why Venn diagrams don't work well to illustrate situations involving 4 or more sets.

10. Write the statement "for every integer x there exists a prime number p such that p divides x" symbolically (use P for the set of prime numbers). Write down its negation symbolically. Write the negation in English.

    Which statement is true?

11. Write the following sentences using quantifier notation.

    (a) There is an integer which is divisible by 2.

    (b) Every integer is divisible by 2.

    (c) Given any integer, there is another integer such that the sum of the two is 0.

    (d) There is some integer such that given any integer the sum of the two is 0.

    Write down the negation of the sentences, both in quantifier notation and in Engish.

12. Prove Proposition 5.4 (i). (Hint: logical equivalence of A and B is the same as saying "A if and only if B.")

13. In an English village, every (adult male) villager who does not shave himself is shaved by the (adult, male) village barber. Let S be the set of villagers who shave themselves, and let B be the set of villagers shaved by the barber.

    Is the barber an element of S or B? Who shaves the barber? Discuss.

14. Let S be the set of all sets which are not elements of themselves. Is S and element of itself, or not? Discuss.