1 Definitions and Theorems

1. Which of the following are true, and which are false?

   (a) $3 \mid 100$.

   (b) $3 \mid 99$.

   (c) $-3 \mid 3$.

   (d) $-5 \mid -5$.

   (e) $-2 \mid -7$

   (f) $0 \mid 4$.

   (g) $4 \mid 0$.

   (h) $0 \mid 0$.

2. A *rational number* is a number formed by dividing two integers a/b where $b \neq 0$. The set of all rational numbers is denoted $\mathbb{Q}$.

   Explain why every integer is a rational, but not every rational is an integer.

3. Define what it means for an integer to be *square*. For example, the integers 0, 1, 4, 9, and 16 are square. Your definition should begin:

   > An integer x is called *square* provided...

4. Suppose the distance between two points in the plane is already defined. Write a careful definition for one point to be between two other points. Your definition should begin:

   > Suppose A, B and C are points in the plane. We say that C is *between* A and B provided...

   Note: when considering your definition, you should decide whether you think that C is "between" A and B if it equals A or B, and make sure that your definition is consistent with this decision.

   Use your definition of "between" to give a definition of what it means for three points to be collinear. Your definition should begin:

   > Suppose A, B and C are points in the plane. We say that they are collinear provided...

   Note: If any of A, B and C are equal, they should be collinear by your definition.

   How does the decision about "between" in the first Note affect the second definition? Which choise do you prefer now?

5. The number 6 has 4 positive divisors, 1, 2, 3, and 6. How many positive divisors do each of the following have (with brief explanations, please)?

   (a) 8.

   (b) 32.

   (c) 2^n where n is a positive integer.

   (d) 10.

   (e) 1000.

   (f) 10^n where n is a positive integer.

   (g) 30 (hint: $30 = 2 \times 3 \times 5$).

   (h) 42 (hint: $30 = 2 \times 3 \times 7$).

   (i) 2310 (hint: $2310 = 2 \times 3 \times 5 \times 7 \times 11$).

   (j) 120 (hint: $120 = 2^3 \times 3 \times 5$).

   (k) $8! = 1 \times 2 \times 3 \times 4 \times 5 \times 6 \times 7 \times 8$.

   (l) 0.

   Briefly explain how you would find the number of positive divisors in general.

6. Rewrite each of the following statements in the form "If A then B."

   (a) "The product of an odd integer and an even integer is even."

   (b) "The square of an odd integer is odd."

   (c) "The square of a prime number is not prime."

   (d) "The product of two negative numbers is negative."

7. It is a common mistake to confuse the two statements "If A then B" and "If B then A." Find two conditions A and B so that the first statement is true, but the second statement is false.

8. Consider the three statements

   (a) If A then B.

   (b) (not A) or B.

   (c) If (not B) then (not A).

   Under which circumstances are these statements true? When are they false? Explain why these statements are, in essence, identical.

9. Consider the rather grotesque claim: “If you pick a guinea pig up by its tail, its eyes pop out.” Is this true?

2 Proofs and Counterexamples

1. Unit NT, Section 1 (p 62), question 1.1.

2. Unit NT, Section 1 (p 62), question 1.2.

3. Unit NT, Section 1 (p 62), question 1.4.

4. Unit NT, Section 1 (p 62), question 1.5.

5. Unit NT, Section 1 (p 63), question 1.14.

6. Let x be an integer. Prove that x is odd if and only if $x + 1$ is even.

7. Let x be an integer. Prove that $0 \mid x$ if and only if $x = 0$.

8. Prove that an integer is odd if and only if it is the sum of two consecutive integers.

9. Disprove: An integer x is positive if and only if $x + 1$ is positive.

10. Disprove: Two right-angled triangles have the same area if and only if the lengths of their hypotenuses are the same.