Harmonic Analysis and Representation Theory of the Automorphism Groups of Homogeneous Trees

An Honours Thesis

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Introduction

This thesis aims to develop some theory about representation theory and harmonic analysis on the automorphism group of a homogeneous tree. To this end, it is divided up into two chapters.

The first details the graph theory, group theory and geometry associated with homogeneous trees that is required for the second chapter. Although trees occur in many areas of mathematics, the first study of homogeneous trees was instituted by Jean-Pierre Serre, and his book *Trees* [11] was the outcome of this. These ideas were added to by P. Cartier, amongst others. Underlying the geometric and graph theoretic aspects, are some concepts from group theory which are not part of a usual introductory course on group theory, so for this reason I have included a section, largely derived from Robinson [10] and Serre [11].

The second chapter starts with some introductory theory on Haar Measures and Representations, and then classifies and discusses two specific types of representations on the automorphism group of a homogeneous tree. The theory of the special and cuspidal representations is essentially due to G. I. Ol'shianskii, with some refinements due to Allessandro Figà-Talamanca and Claudio Nebbia. However in the treatment of the cuspidal representations I have attempted to highlight the similarities between these and the special representations, at the cost of glossing over some of the technical details.

At the end of each chapter I have included a short section which gives some additional information, without proof, about how the substance of the chapter might be generalised, or important extensions to the theory.

The source for most of this thesis has been A. Figà-Talamanca and C. Nebbia, *Harmonic Analysis and Representation Theory for Groups Acting on Homogeneous Trees* [4], and most of the results and proofs come from here, although many have been altered to some extent. I have been slightly more thourough at the basic levels, providing proofs for a number of results that are simply stated in Figà-Talamanca, and going into more detail in some of the other areas, such as the discussion of the point stabiliser subgroups. The only other major difference is the method of showing unimodularity of the automorphism group. I have approached though the expression of the automorphism group as a product of groups, whilst Figà-Talamanca uses the concept of Gelfand pairs to get the same result.

Chapter 1

Automorphisms of Trees

In this chapter we build up some graph and group theory and then investigate the automorphisms of trees and the structure of the group of automorphisms of a tree. We basically follow Figà-Talamanca [4] throughout this chapter, however Section 1.2 introduces some ideas from group theory which are largely from Robinson [10], and occasional ideas come from Serre [11].

1.1 Graphs and Trees

Before we proceed, we need some basic graph theory.

Definition 1.1.1 *We make the following definitions:*

An oriented graph $\Gamma = (V, E)$ *consists of a set* V*, of* vertices *of* Γ*, a set* E*, of* (directed) edges *of* Γ*, and an* incidence relation *which is a map*

$$y \mapsto (o(y), t(y)) : E \to V \times V$$

For $y \in Y$ *we call* $o(y)$ *the* origin *of* y*,* $t(y)$ *the* terminus *of* y*, and together they are the* extremities *of* y*.*

If the incidence relation is injective, then we can denote each $y \in Y$ *uniquely as an ordered pair* $(u, v) \in V \times V$*, where* $u = o(y)$ *and* $v = t(y)$*.*

If, in addition, an oriented graph $\Gamma = (V, E)$ *has a map*

$$y \mapsto \overline{y} : E \to E$$

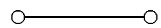
such that for each $y \in Y$ *we have* $\overline{\overline{y}} = y$*,* $\overline{y} \neq y$ *and* $o(y) = t(\overline{y})$*, then* Γ *is referred to as a* (non-directed) graph*.*

For $y \in Y$ *we call,* $\overline{y}$ *the* inverse *of* y*, the pair* $\{y, \overline{y}\}$ *a* (non-directed) edge *of* Γ*. Again, we refer to* $o(y) = t(\overline{y})$ *and* $t(y) = o(\overline{y})$ *as the* extremities *of the edge* $\{y, \overline{y}\}$*.*

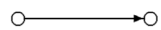
If the incidence relation is injective, then we can denote every non-directed edge $\{y, \overline{y}\}$ *uniquely by the unordered pair* $\{u, v\}$*, where* $u, v \in V$ *and* u*,* v *are the extremities of* $\{y, \overline{y}\}$*.*

Some references refer to graphs where the incidence relation is not injective as *multi-graphs.*

Graphs can be represented diagrammatically by marking a point for each vertex of the graph, and a line for each pair of edges $\{y, \overline{y}\}$, with endpoints of the line being the points corresponding to the vertices which are the extremities of the edge, like this:



In a similar fashion, we can draw a diagram representing a directed graph, but we represent the directed edge y by a line with an arrow pointing from $o(y)$ to $t(y)$, like this:



Definition 1.1.2 *Let $\Gamma = (V, E)$ be a (possibly directed) graph.*

A pair of distinct vertices are adjacent *if they are extremities of the same edge.*

The valence *or* degree *of a vertex $v \in V$, $\mathrm{val}(v)$, is the number of directed edges $y \in E$ for which $o(y) = v$. If Γ is non-directed, then this equals the number of directed edges $y \in E$ for which $t(y) = v$. The valence of a vertex may be infinite.*

A graph is locally finite *if the valence of every vertex is finite.*

A path *in Γ is an alternating sequence of vertices and edges, $v_0, y_0, v_1, y_1, \ldots, y_{n-1}, v_n$, where $v_i \in V$, $y_i \in E$ and $o(y_i) = v_i$ and $t(y_i) = v_{i+1}$, for all i.*

The length *of a path with n edges, $v_0, y_0, v_1, y_1, \ldots, y_{n-1}, v_n$, is n.*

A graph Γ is said to be connected *if all pairs of distinct vertices have a path between them.*

A chain *in Γ is a path, $v_0, y_0, v_1, y_1, \ldots, y_{n-1}, v_n$, for which $y_i \neq \overline{y_{i+1}}$ for all $i \in \{0, \ldots, n-2\}$.*

A loop *in Γ is a chain, $v_0, y_0, v_1, y_1, \ldots, y_{n-1}, v_n$, where $v_0 = v_n$*

Note that if the incidence relation is injective, then we can refer to a path simply by the vertices, as there is a unique edge defined by the extremities. In this case a path is a chain if $v_i \neq v_{i+2}$, for all $i \in \{0, \ldots, n-2\}$.

We have now reached the point where we can define what a tree actually is.

Definition 1.1.3 *A* tree *is a connected, non-directed graph with no loops.*

A direct consequence of this definition is that given a tree $\Gamma = (V, E)$, every pair of points $u, v \in V$ has a unique chain between them (existence follows from the connectedness of the graph, and uniqueness from the fact that there are no loops). We call this unique chain the *geodesic* between u and v, and we represent it symbolically as $[u, v]$.

The following are the two types of trees which will be of special interest.

Figure 1.1: Homogeneous trees

Definition 1.1.4 *A* homogeneous *tree of order* $q+1$ *is a locally finite tree where each vertex has valence* $q+1$.

A semi-homogeneous *tree of order* $(q+1, r+1)$ *is a locally finite tree where each vertex has either a valence of* $q+1$ *or a valence of* $r+1$*, and no two vertices of the same order are adjacent.*

(The convention of using $q+1$ and $r+1$ springs from convenience in handling spherical functions)

One could conceive of a homogeneous tree of order 0 as being a single vertex with no edges, however this is a trivial case and we will ignore it. One must also note that a semi-homogeneous tree of order $(q+1, q+1)$ is in fact a homogeneous tree of order $q+1$. The first few examples of each type of tree are illustrated in figures 1.1 and 1.2.

Having defined the objects with which this thesis is concerned, we will now define some further useful concepts from graph theory:

Definition 1.1.5 *Two graphs* $\Gamma = (V, E)$ *and* $\Lambda = (U, D)$ *are* isomorphic *if there exists a bijective map* $\alpha : \Gamma \to \Lambda$ *such that* α *maps vertices to vertices, edges to edges and preserves the incidence relations, ie. if* i *and* j *are the respective incidence relations, then* $i = j \circ \alpha$.

The map α *is called a* (graph) isomorphism.

Definition 1.1.6 *Let* $\Gamma = (V, E)$ *be a (possibly directed) graph.* (U, D) *is a* subgraph *of* Γ *if* $U \subseteq V$, $D \subseteq E$ *such that for all* $y \in D$, $o(y) \in U$ *and* $t(y) \in U$.

A subtree *of a graph is a subgraph which is also a tree.*

A subtree $T = (U, D)$ *is* complete *if for all* $u \in U$*, the valence of* u *in* T *is either 0, 1 or the same as the valence of* u *in* Γ*. (The 0 valence case only applies for the trivial tree* $U = \{u\}$*.)*

Figure 1.2: Semi-homogeneous trees

Note that since a tree has no loops, a subgraph of a tree is automatically a subtree if it is connected and non-directed.

There is nothing unusual about the above definitions, and they are almost exactly what one would expect.

Definition 1.1.7 *Let $\Gamma = (V, E)$ be a non-directed graph.*

An orientation *of Γ is a subset E_+ of E, such that E is the disjoint union $E_+ \sqcup \overline{E}_+$.*

An orientation can be considered a way of getting a pair of directed subgraphs from a non-directed graph, by setting $\Gamma_+ = (V, E_+)$ and $\overline{\Gamma}_+ = (V, \overline{E}_+)$.

Definition 1.1.8 *Let $\Gamma = (V, E)$ be a (possibly directed) graph. The* barycentric subdivision *of Γ is the graph $\Gamma' = (V', E')$ where $V' = V \cup E$ and $E' = E \times \{0, 1\}$, with the incidence relation*

$$(y, i) \mapsto (o'(y, i), t'(y, i))$$

defined by

$$o'(y, 0) = o(y)$$
$$t'(y, 0) = y = o'(y, 1)$$
$$t'(y, 1) = t(y)$$

Intuitively, the effect of barycentric subdivision is to create a new vertex in the "middle" of each edge. (See figure 1.3).

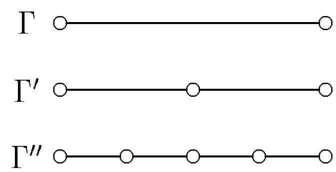


Figure 1.3: Barycentric Subdivision

1.2 Concepts from Group Theory

In this section we will explore some ideas from group theory, essentially following Robinson [10], and make some connections to the theory of graphs.

Definition 1.2.1 *Given a group G, and a subset X of G, we define the subgroup* generated *by X, written $< X >$, as the subgroup which is the intersection of all subgroups which contain X. We also define the* normal closure *of X in G to be the normal subgroup which is the intersection of all normal subgroups which contain X.*

We notice that $< X >$ is the set of all elements of the form $x_0^{\varepsilon_0} \dots x_k^{\varepsilon_k}$ where $\varepsilon_i \in \{-1, +1\}$, $x_i \in X$ and $k \geq 0$ (if $k = 0$, then we treat the product as being 1_G). This leads us to the following definitions.

Definition 1.2.2 *Given an arbitrary set S, called an* alphabet *of* symbols*, we define:*

A word *on S is a finite sequence symbols of S, $(s_1, \dots, s_n)$, usually written $s_1 \dots s_n$, with $s_i \in S \cup S^{-1}$ for $1 \leq i \leq n$, where S^{-1} is the set $\{t^{-1} : t \in S\}$.*

A reduced word *on S is word where no $s \in S$ is adjacent to its inverse $s^{-1} \in S^{-1}$, that is, there are no occurrences of ss^{-1} or $s^{-1}s$ for any $s \in S$.*

The product *of two words $s_1 \dots s_n$ and $t_1 \dots t_m$ is the concatenation of the two words, ie. the word $s_1 \dots s_n t_1 \dots t_m$.*

The inverse *of a word $s_1 \dots s_n$ is the word written in the reverse order with every term inverted, ie. the word $s_n^{-1} \dots s_1^{-1}$ (we define $(s^{-1})^{-1} = s$).*

The empty word *e is the word with no symbols.*

We note that every word can be turned into a unique reduced word by repeatedly removing all adjacent pairs of symbols and their inverses. We can therefore define concatenation on reduced words by normal concatenation, followed by reduction. It it is easy to see that for set X, the set of reduced words on that set forms a group under this concatenation, with the inverse being the word inverse, and e the identity of the group.

Definition 1.2.3 *Given a set X, the group F of reduced words on X is called the* free group *on X.*

We note that $F = < X >$, and we will often write $< X >$ for the free group on X.

An important result of concerning free groups is that given a set X and a map ϕ from X to some group G, there is a homomorphism α from $< X > \to G$ such that given the map β from X into $< X >$ defined by identifying X with the appropriate generating element of $< X >$, $\phi = \alpha \circ \beta$. This is called the universal property of free groups.

A corollary of this is that is that every group is the image of a free group under some homomorphism, that is, given a group G, there exists a free group F and an epimorphism (surjective homomorphism) $P : F \to G$. This is derived from the universal property by setting $X = G$, and getting $F = < G >$.

Definition 1.2.4 *Such an epimorphism $P : F \to G$ is called a* (free) presentation *of G.*

The normal subgroup of F, $R = \ker(P)$, is called the set of relators *of the presentation.*

If S is a subset of F such that the normal closure of S in F is R, then we call S a set of defining relators *and all $r \in R \setminus S$ are called* consequences *of S.*

Given a projection $P : F \to G$, if $F = < X >$, then $Y = P(X)$ generates G. If S is a set of defining relators, then we will write $G = < X \mid S >$, since G is isomorphic to the group generated by G with the normal closure of S factored out. Sometimes we will simply refer to $< X \mid S >$ as a presentation of G. We may also write this as $< X \mid s = e : s \in S >$.

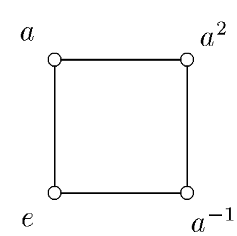
We now use presentations to provide a link between groups and graphs.

Definition 1.2.5 *Given a group G with a presentation P such that $G =< X \mid R >$, we define the* Cayley graph $\Gamma(G, P)$ *of the presentation as the graph $\Gamma = (G, E)$, where for all $g, h \in G$, (g, h) is an edge if $\exists\ x \in X$ such that $gx = h$ or $gx^{-1} = h$.*

We notice immediately that the Cayley graph must be connected and that if the number of generators is finite, then the valence of every vertex of Γ must be constant.

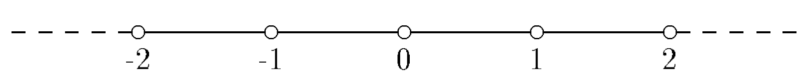
Example 1.2.1 *The cyclic group C_4 over the symbol a has a presentation $< a \mid a^4 = e >$. So the edge set of the Cayley graph is:*

$$\{\{e, a\}, \{a, a^2\}, \{a^2, a^{-1}\}, \{a^{-1}, e\}\},$$



Example 1.2.2 *The group* **Z** *has a presentation $< 1 >$. So the edge set of the Cayley graph is:*

$$\{\ldots, \{-2, -1\}, \{-1, 0\}, \{0, 1\}, \{1, 2\}, \ldots\}$$



We note that the Cayley graph of Example 1.2.2 is a tree, and as we are interested in trees, we want to know if there are other groups which have Cayley graphs which are trees.

Proposition 1.2.1 *The Cayley graph of a free group, F, is a tree.*

Proof:

We know that the Cayley graph of F is connected, so it only remains to be shown that it has no loops.

Assume that there is a loop in the Cayley graph of F. That is, there is a chain $\gamma = x_0, \ldots, x_n$, with $x_0 = x_n$. This implies that there is a reduced word, W, of generators of F of length n such that $x_0 W = x_n = x_0$, that is $W = e$. But by the definition F, there is no such word, other than the empty word, e, which has length $n = 0$, and no chain can have length $n = 0$. So we have a contradiction. ■

All these trees have an even valence for their vertices. There are groups which have Cayley graphs which are trees where all the vertices have odd valence, but to describe these we need the concept of a free product.

Definition 1.2.6 *Given a family of (disjoint) groups, $\{G_i\}_{i \in I}$ for some index set I, we define the* free product *of the G_i to be*

$$\mathop{\ast}_{i \in I} G_i = < \bigcup_{i \in I} G_i \mid S >$$

where

$$S = \{g_{i_1} g_{i_2} g_{i_3}^{-1} : g_{i_1}, g_{i_2} \in G_i \text{ for some } i \in I, \ g_{i_3} = g_{i_1} g_{i_2}\}$$

On an intuitive level, this construction is imposing the multiplication of each group on all appropriate elements of the free group generated by the G_i, since what we are doing is in effect setting $g_{i_1} g_{i_2} g_{i_3}^{-1} = e$, ie. $g_{i_1} g_{i_2} = g_{i_3}$.

More concretely, it can be shown (Robinson [10, §6.2, p163]) that the elements of the free product are exactly the reduced words of the form $g_1 g_2 \ldots g_3$ where if $g_i \in G_j$, then $g_{i+1} \notin G_j$ and with multiplication being concatenation of the words, followed by reduction via

If $g_i, g_{i+1} \in G_j$ for some j, then replace them by their product in G_j, $g_i g_{i+1}$.

If $g_i = e_j$, the identity element of G_j, then remove g_i.

Repeat these steps as necessary.

Example 1.2.3 *Let $G_1 = G_2 = \mathbf{Z}_2$, then if a is a generator of G_1, and b is a generator of G_2, then*

$$G_1 * G_2 =< \{a, b\} \mid a^2 = b^2 = e >$$

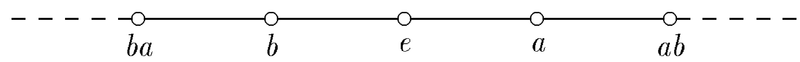
or, using the reduced word form,

$$G_1 * G_2 = \{\ldots, baba, bab, ba, b, e, a, ab, aba, abab, \ldots\}$$

The next obvious step is to look at the Cayley graph of this group. We get the edge set:

$$\{\ldots, \{ba, b\}, \{b, e\}, \{e, a\}, \{a, ab\}, \ldots\}$$

which has the following diagram:



We note that the graph of this group and that of the integers are isomorphic.

Example 1.2.4 *Let $G_i = \mathbf{Z}$ for $i = 1, \ldots, n$. Then*

$$
\begin{aligned}
\mathop{\ast}_{i=1}^{n} \mathbf{Z} &= < \{a_i^j : i = 1, \ldots, n,\ j \in \mathbf{Z}\} \mid \{a_i^j a_i^k a_i^{-j-k} : i = 1, \ldots, n,\ j, k \in \mathbf{Z}\} > \\
&= < \{a_i : i = 1, \ldots, n\} > \\
&= F_n
\end{aligned}
$$

So the free product of n copies of the integers is F_n. In fact $F_n * F_m = F_{n+m}$.

We now have enough to describe a large class of groups for which the Cayley graphs are trees.

Theorem 1.2.2 *The Cayley graph of*

$$F_s * \underbrace{\mathbf{Z}_2 * \cdots * \mathbf{Z}_2}_{t \text{ times}}$$

is a homogeneous tree of order $q + 1$ where $q + 1 = 2s + t$.

Proof:

The graph is a Cayley graph, so it is connected and the valence of each vertex is constant. So we need to show that there are no loops.

Let $X = \{a_1, \ldots, a_s, b_1, \ldots, b_s\}$ be the set of generators, then if there is a loop in the Cayley tree, $\gamma = v_0, \ldots, v_n$, such that $v_0 = v_n$, then there must be a reduced word W on X such that $v_0 W = v_n = v_0$. Thus $W = [e]$. W cannot be $a_i a_i^{-1}$ or $b_j^2 = b_j b_j^{-1}$, since both these imply that two consecutive edges in the loop are inverses of each other. Thus $W = e$.

Thus there are no loops, so the Cayley graph is a homogeneous tree.

Finally, there are $2(t + s)$ possible directed edges out of e, corresponding to each element of $X \cup X^{-1}$, but since $b_i = b_i^{-1}$, s of these have been double-counted. Thus there are $2t + s$ edges from e, so $q + 1 = 2t + s$. ■

It turns out that these are the only groups which have locally finite Cayley graphs which are trees (see section 1.8).

We now discuss a generalisation of the free product, the amalgamated product.

Definition 1.2.7 *Given a family of (disjoint) groups, $\{G_i\}_{i \in I}$ for some index set I, a group A, and a collection of injective homomorphisms $\phi_i : A \rightarrow G_i$, for all $i \in I$ (that is we can think of A as a subgroup of G_i for all i). Then we define the* amalgamated product *of the G_i to be*

$$\mathop{\ast}_{A} G_i = \frac{\mathop{\ast}_{i \in I} G_i}{N}$$

where N is the normal closure of the set

$$\{\phi_i(a)\phi_j(a)^{-1} : i, j \in I,\ a \in A\}$$

in $\mathop{\ast}\limits_{i\in I} G_i$.

Intuitively, this is "amalgamating" all the "copies" of A in the group $\mathop{\ast}\limits_{i\in I} G_i$ into just one "copy".

More concretely, it can be shown (Serre [11, §1.2, p2]), that the elements of $\mathop{\ast}\limits_{A} G_i$ can be represented as pairs $(a, s_1 \dots s_n)$, where $a \in A$ and $s_i \in S_j \setminus \{1\}$, S_j a set of right coset representatives of $\phi_j(A)$ in G_j, and if $s_i \in S_j \setminus \{1\}$ then $s_{i+1} \notin S_j \setminus \{1\}$.

To conclude this section we look at a generalisation of the direct product, the semidirect product.

Definition 1.2.8 *Given a group G, $N \lhd G$, and $H \leq G$ such that $G = HN$ and $H \cap N = 1$, then G is said to be the* internal semidirect product *of N and H, written $G = H \rhd\!\!< N$ or $G = N >\!\!\lhd H$.*

From Robinson [10, §1.5, p27], we get that each element of G has a unique expression of the form hn $(h \in H,\ n \in N)$ and that conjugation of N by an element of H gives an automorphism of N, so we get a homomorphism $\alpha : H \to \mathrm{Aut}(N)$ (in other words H *acts* on N). We note that if this homomorphism is the trivial homomorphism, then $G = H \times N$.

These facts allow us to reverse the definition and construct a G from H and N.

Definition 1.2.9 *Given two groups H and N and a homomorphism $\alpha : H \to \mathrm{Aut}(N)$, we define the* (external) semidirect product *of H and N, written $H \rhd\!\!<_\alpha N$ or $N >\!\!\lhd_\alpha H$, as the set $\{(h, n)\ :\ h \in H,\ n \in N\}$ with multiplication defined by*

$$(h_1, n_1)(h_2, n_2) = (h_1 h_2, \alpha(h_2)(n_1)n_2)$$

We notice again that if α is the trivial homomorphism, then the semidirect product and the direct product coincide. The definition of the multiplication is motivated by the fact that we are trying to simulate conjugation by h_2 and the fact that $(h_1 n_1)(h_2 n_2) = h_1 h_2 (h_2^{-1} n_1 h_2) n_2$ always.

Robinson shows that if $G = H \rhd\!\!<_\alpha N$ then G is also the internal semidirect product of H and N when imbedded in G by $h \mapsto (h, e)$ and $n \mapsto (e, n)$ respectively. Thus we will, in future, simply refer to the semidirect product of two groups.

1.3 Metrics on Trees

One can define a metric d_V on the vertices of a tree (V, E) in the following way:

Definition 1.3.1 *The* distance $d_V(v, u)$ *between two vertices $v, u \in V$ is the length of the unique chain between v and u, or 0 if $v = u$.*

Proof:

d_V is by definition a map from V to $\{0, 1, 2, \ldots\}$, so it certainly has the correct range for a metric.

By definition, $d_V(v, v) = 0$.

If $d_V(v, u) = 0$ then since all chains have a length of at least 1, there cannot be a chain between v and u, so they must be the same vertex.

If $d_V(v, u) = n$, then there is a chain $v, v_1, \ldots, v_{n-1}, u$, but since $(v_i, v_{i+1}) \in E$ implies $(v_{i+1}, v_i) \in E$, $u, v_{n-1}, \ldots, v_1, v$ is a chain, and thus it must be the unique chain between u and v. $u, v_{n-1}, \ldots, v_1, v$ has length n, so $d_V(u, v) = d_V(u, v)$.

If $d_V(v, u) = n$, and $d_V(u, w) = m$ then

$$\begin{aligned} d_V(v,u) + d_V(u,w) &= \text{length of } v, v_1, \ldots, v_{n-1}, u + \text{length of } u, u_1, \ldots, u_{m-1}, w \\ &= \text{length of } v, v_1, \ldots, v_{n-1}, u, u_1, \ldots, u_{m-1}, w \\ &\geq \text{length of } v, v_1, \ldots, v_{n-a}, u_a, \ldots, u_{m-1}, w \\ &\geq d_V(v, w) \end{aligned}$$

So d_V is a metric on V. ■

The metric d_V on V gives sufficient information to construct the whole tree, since E is exactly the set of all pairs $\{v, u\}$ where $d_V(v, u) = 1$.

Definition 1.3.2 *Given some vertex $v \in V$, we define the* ball *of radius r at v by*

$$B_v(r) = \{u \in V : d_V(v, u) \leq r\}$$

and the boundary *of $B_v(r)$ as*

$$\partial B_v(r) = \{u \in V : d_V(v, u) = r\}$$

More generally, given a finite subtree $T = (U, D)$, we define the boundary of T by

$$\partial T = \{u \in U : \mathrm{val}_T(u) < val_\Gamma(u)\}$$

Finally, for a subtree T, we define the diameter *of T to be*

$$diam(T) = max\{d_V(u, v) : u, v \in T\}$$

We will sometimes abuse notation by referring to the subtree $(B_v(r), D)$, where D is the set of edges with both extremities in $B_v(r)$ as $B_v(r)$. In this case the two definitions of the boundary coincide. We will extend the definition of boundaries to infinite trees in section 1.4.

The metric topology on V, (V, d_V) is the discrete topology on V, and it is well known that we can construct a metric d_C on the compact sets of this topology in the following manner:

Definition 1.3.3 *Given two compact (ie. finite, since (d_V, V) discrete) sets $A, B \subseteq V$, define*

$$A_r = \{v \in V : d_V(v, u) \leq r, \textit{ for some } u \in A\}$$

and B_r similarly, then

$$d_C(A, B) = \inf\{r \in \mathbf{Z} : A \subseteq B_r and B \subseteq A_r\}$$

In particular we note that since finite subtrees and edges can be represented as finite subsets of V, we can define metrics on the set of subtrees and on the set of edges as follows:

Definition 1.3.4 *Given two finite subtrees* $T_1 = (U_1, D_1)$ *and* $T_2 = (U_2, D_2)$ *of a tree, then the distance between them is defined to be* $d_T(T_1, T_2) = d_C(U_1, U_2)$.

Given two edges $y_1 = \{u_1, v_1\}$ *and* $y_2 = \{u_2, v_2\}$ *of a tree, we define the distance between the edges to be* $d_E(y_1, y_2) = d_C(\{u_1, v_1\}, \{u_2, v_2\})$.

In effect these two definitions are simply the restriction of the compact metric to the set of finite trees and the set of edges respectively. In all three cases the metric topology will again be the discrete topology on the appropriate set.

1.4 The Boundary of an Infinite Tree

From Section 1.1, we note that once the order reaches a certain size, (specifically when all arguments of the order are ≥ 2), both the homogeneous and semi-homogeneous trees become infinite. This leads to the possibility of having chains of infinite length and additional structure derived from them. However, the following discussion is not limited to homogeneous trees, but in fact holds for any infinite tree.

Definition 1.4.1 *An* infinite chain *is an infinite sequence* $v_0, v_1, v_2, \ldots$ *of vertices such that for all* i, $\{v_i, v_{i+1}\}$ *is an edge and* $v_i \neq v_{i+2}$.

Two chains $v_0, v_1, v_2, \ldots$ *and* $y_0, y_1, y_2, \ldots$ *are* equivalent *if there is some* n *and some* m *for which* $v_{n+k} = u_{m+k}, \ \forall\ k \geq 0$.

The boundary, Ω, *of an infinite tree is the set of equivalence classes of infinite chains.*

If the infinite chain $v_0, v_1, v_2, \ldots$ *is in the equivalence class* $\omega \in \Omega$ *then we say* ω *is an* endpoint *of* $v_0, v_1, v_2, \ldots$.

Sometimes it will be convenient to think of the boundary of a tree as being the set of all infinite chains of the form $o, v_0, v_1, v_2, \ldots$ where o is some fixed vertex. These two concepts are equivalent, since the chains $o, v_0, v_1, v_2, \ldots$ are merely a complete set of representatives from each equivalence class of Ω.

Definition 1.4.2 *A* doubly infinite chain *is a two-sided infinite sequence of vertices,* $\ldots, v_{-2}, v_{-1}, v_0, v_1, v_2, \ldots$ *such that for all* i, $\{v_i, v_{i+1}\}$ *is an edge and* $v_i \neq v_{i+2}$.

We now want to extend the definition of geodesics to include the boundary.

Definition 1.4.3 *We call the infinite chain* $v_0, v_1, v_2, \ldots \in \omega$ *the* (infinite) geodesic *from* v_0 *to* ω, *written* $[v, \omega)$, *and the doubly infinite chain* $\ldots, v_{-2}, v_{-1}, v_0, v_1, v_2, \ldots$ *with* $v_0, v_1, v_2, \ldots \in \omega_+$ *and* $v_0, v_{-1}, \ldots \in \omega_-$ *the* (infinite) geodesic *from* ω_- *to* ω_+, *written* (ω_-, ω_+).

For any given $v \in V$, $\omega, \omega_+, \omega_- \in \Omega$, the infinite geodesics $[x, \omega)$ and (ω_-, ω_+) are unique.

We can now define a topology on $V \cup \Omega$ as follows.

Figure 1.4: An Open Neighbourhood of ω

Definition 1.4.4 *We define the topology by setting each element of* V *open, and define a basis of open neighbourhoods for each element of* Ω.

Let $v \in V$, $\omega \in \Omega$. *The for each* $u \in [v, \omega)$ *we define the neighbourhood* $\mathcal{N}_\omega(v, u)$ *of* ω *to be the set of all vertices and end points of the infinite chains* $[x, \nu)$ *such that* $[v, u] \cap [x, \nu) = \{u\}$ *(ie. the infinite chains which contain no vertex of* $[v, u]$ *except* u*).*

See Figure 1.4 for a diagram of what the open neighbourhoods look like. Note that there is considerable duplication amongst the neighbourhoods: If $[v, \omega) \cap [u, \omega) = [x, \omega)$ then $\forall\ y \in [x, \omega)$, $\mathcal{N}_\omega(v, y) = \mathcal{N}_\omega(u, y)$, and if $[v, u] \subseteq [v, \omega) \cap [v, \nu)$ then $\mathcal{N}_\omega(v, u) = \mathcal{N}_\nu(v, u)$.

Theorem 1.4.1 $V \cup \Omega$ *is compact.*

Proof:

Choose some vertex $o \in V$. Let $a_r = max\{\text{val}(v) : v \in \partial B_o(r)\}$. Let $\Lambda = (U, D)$ be a tree for which there is some $o \in U$ such that $\text{val}(v) = a_{d_U(v,o)}$ for all $v \in U$. Clearly Γ is isomorphic to a subtree of Λ. Let Δ be the boundary of Λ. We will show $U \cup \Delta$ is compact.

Firstly we can identify Δ with $\Pi = \mathbf{Z}_{a_0} \times \prod_{i=1}^{\infty} \mathbf{Z}_{a_i - 1}$ since the chain from o to $\delta \in \Delta$ has a_0 choices for the second element, and $a_i - 1$ for the $(i+1)$st element of the chain. Furthermore this map is a homeomorphism since the product topology on Π has a base of neighbourhoods where given $\gamma \in \Pi$, $n \in \mathbf{Z}$, we have the neighbourhood $\mathcal{N}_\gamma(n) = \{\lambda \in \Pi : \lambda_i = \gamma_i,\ 0 \le i \le n\}$, which exactly the corresponds with restriction of the open neighbourhoods of $U \cup \Delta$ to Δ. Thus Δ is compact, since by Tychanoff's theorem, Π is compact.

Let $\{W_i\}$, $i \in I$ be an open cover of basic sets of $U \cup \Delta$. Then $\{W_i \cap \Delta\}$, $i \in I$ is an open cover of Δ. Therefore there is a finite subcover $\{W_j \cap \Delta\}_{j=1}^{m}$ of Δ.

Figure 1.5: “Pruning” Λ

Now there must be some m such that

$$\partial B_o(r) \subseteq \bigcup_{j=1}^{l} W_j$$

otherwise we would be able to construct an element of Δ which was not covered by $\{W_j \cap \Delta\}_{j=1}^{m}$.

So $(U \cup \Delta) \setminus (\bigcup_{j=1} lW_j) \subseteq B_o(r)$. Now for each $v \in B_o(r)$, there exists some $W_k,\ k \in I$ such that $v \in U_k$. Since $|B_o(r)|$ is finite, we can append this finite number of additional sets to $\{W_j\}_{j=1}^{m}$ and get a finite subcover of the whole set.

Thus $U \cup \Delta$ is compact. This means that $V \cup \Omega$ is compact, since we can get $V \cup \Omega$ from $U \cup \Delta$ by “pruning” Λ down to Γ. Each time a vertex is removed from Λ, an open neighbourhood is removed from $U \cup \Delta$. (See figure 1.5). Thus $V \cup \Omega$ is closed, since it is homeomorphic to $U \cup \Delta$ minus some union of open sets. So $V \cup \Omega$ is compact. ■

Corollary 1.4.2 *Ω is compact.*

Proof:
$V \cup \Omega$ is compact. V is a union of vertices, which are all open, so V is open. $\Omega = (V \cup \Omega) \setminus V$. Thus Ω is closed, so Ω is compact. ■

Proposition 1.4.3 *V is dense in $V \cup \Omega$.*

Proof:
Consider the geodesic $[v_0, \omega)$. The sequence $\{v_i\}_{i=1}^{\infty}$ clearly converges to ω. For each $\omega \in \Omega$ there exists such a sequence, so we have our result. ■

We can also define a collection of measures on Ω.

Definition 1.4.5

$$\Omega(u,v) = \{\omega \in \Omega : [u,v] \subset [u,\omega)\}$$

Then $\{\Omega(u,v) : v \in \partial B_u n\}$ partitions Ω into $|\partial B_u n|$ open and compact sets. We can then use these partitions to generate a σ-algebra on Ω.

Definition 1.4.6 *Given a path $v_0, \ldots, v_n$, we define the following function*

$$p_{v_0}(v_n) = \mathrm{val}(v_0) \times \prod_{i=1}^{n-1} \mathrm{val}(v_i)$$

$\frac{1}{p_{v_0}(v_n)}$ is the probability that a random walk starting at v_0 will pass through v_n. We use this function to define a probability measure ν_u on the σ-algebra above, by setting $\nu_u(\Omega(u,v)) = \frac{1}{p_u(v)}$. Since this is a probability measure, $\nu_u(\Omega) = 1$.

Furthermore, since the $\Omega(u,v)$ are a basis for the relative topology on Ω, we can extend the σ-algebra to the Borel σ-algebra, and the measure to a Borel measure.

1.5 Automorphisms of Trees

Now, following Figà-Talamanca [4], we consider mappings of trees onto themselves.

Definition 1.5.1 *An* automorphism *of a tree (V,E) is a bijective map $g : V \to V$ which preserves edges. ie. $g(u)$ and $g(v)$ are adjacent iff u and v are adjacent.*

These automorphisms are exactly the surjective isometries of the metric space (V, d_V).

Lemma 1.5.1 *g is an automorphism of (V,E) iff g is an isometry of (V, d_V).*

Proof:

If g is an isometry of (V, d_V), then $g(v)$ and $g(u)$ are adjacent iff $d_V(g(v), g(u)) = 1$ iff $d_V(v,u) = 1$ (since g is an isometry) iff v and u are adjacent. So g is an automorphism of (V,E).

Conversely, if g is an automorphism of (V,E), then if $d_V(v,u) = \mathrm{n}$, there exists a unique chain $v, v_1, \ldots, v_{n-1}, u$ of length n. But $g(v), g(v_1), \ldots g(v_{n-1}), u$ is a chain, as for all i, $g(v_i)$ and $g(v_{i+1})$ are adjacent since v_i and v_{i+1} are adjacent, and $g(v_i) \neq g(v_{i+1})$ since g is a bijection. $g(v), g(v_1), \ldots g(v_{n-1}), u$ has length n, so $d_V(g(v), g(u)) = n$. So g is an isometry of (V, d_V). ■

We can now split automorphisms of a tree into three classes.

Definition 1.5.2 *An automorphism g of (V,E) is:*

i. A rotation *about o if g stabilises some vertex o, ie. $\exists\ o \in V$ such that $g(o) = o$.*

ii. An inversion *about an edge e if g stabilises e, but exchanges its endpoints.*

iii. A translation *of* step *j along a geodesic γ if there is a geodesic*

$$\gamma = \ldots, v_{-2}, v_{-1}, v_0, v_1, v_2, \ldots$$

such that $g(v_n) = v_{n+j}$ for all n.

Theorem 1.5.2 *For all automorphisms g of a tree (V, E), g is either a rotation, an inversion or a translation.*

Proof:

Let $v \in V$ be a vertex such that $d_V(v, g(v)) = \min\{d_V(u, g(u)) : u \in V\} = j$.

If $j = 0$, then $g(v) = v$, so g is a rotation.

If $j = 1$, then if $g^2(v) = v$, then g interchanges v and another adjacent vertex, so g must be an inversion. If $g^2(v) \neq v$ then let γ be a geodesic,

$$\gamma = \ldots, v_{-2}, v_{-1}, v_0, v_1, v_2, \ldots,$$

where $v_n = g^n(v)$. Then $g(g^n(v)) = g^{n+1}(v)$, so g is a translation of step 1 along γ.

If $j \geq 2$ then let $[v, g(v)] = v, v_1, \ldots, v_{j-1}, g(v)$. We note that $g(v_1) \neq v_{j-1}$, since $d_V(v_1, v_{j-1}) = j - 2 < j$. We can now extend $[v, g(v)]$ by letting $v_{j+k} = g(v_k)$ for all k. v_j and v_{j+1} are adjacent for all n since g is an automorphism, and $v_j \neq v_{j+2}$ since g is bijective, so

$$\gamma = \ldots, v_{-2}, v_{-1}, v_0, v_1, v_2, \ldots, v_{j-1}, g(v), v_{j+1}, \ldots$$

is a geodesic for which $g(v_n) = v_{n+j}$. So g is a translation of step j along γ.

So we have our result. ■

Note another result undelies the case $j \geq 2$.

Lemma 1.5.3 *Let g be an automorphism, and v a vertex. Let*

$$v, v_1, \ldots, v_{n-1}, g(v) = [v, g(v)]$$

be the chain joining v and $g(v)$. If $g(v_1) \neq v_{n-1}$, then g is a translation of step n along a doubly infinite chain of which $[v, g(v)]$ is a subchain.

Proof:

This is proved by extending the chain $[v, g(v)]$ exactly as for the case $j \geq 2$ above. For a full proof, see Figà-Talamanca [4, I, §3, p10]. ■

For a homogeneous tree of order ≥ 2 there are always all three types of automorphisms, however this is not the case for other types of trees. Any finite tree cannot have translations, since there are no infinite geodesics, thus the only possible automorphisms of a finite tree are rotations and inversions. Also, for any automorphism, g, the valence of v and the valence of $g(v)$ must be the same, so in any tree for which there are no two adjacent vertices with the same valence, there are no inversions, and all translations must have step ≥ 2. So a semi-homogeneous tree has only rotations and translations of even step, since translations of odd step would map vertices of different valencies onto each other.

It is also worthwhile noting that the type of automorphism is preserved under conjugacy, ie. $gg'g^{-1}$ is of the same type as g'.

Also, clearly if g is a translation of step j along a geodesic γ, then g^n is a translation of step $|nj|$ along γ.

Proposition 1.5.4 *Given a tree, $\Gamma = (V, E)$, we have:*

i. The composition of two inversions on distinct edges is a translation of even step.

Figure 1.6: Diagram for Proposition

ii. *The composition of an inversion about an edge and a rotation which does not fix both vertices of the edge is a translation of odd step on a geodesic containing that edge.*

Proof:

i. Let h and g be inversions on the distinct edges $x = \{v, u\}$ and $y = \{w, z\}$, ie. $d_E(x, y) \neq 0$. See Figure 1.6.

   If $d_E(x, y) = 1$, then the two edges have 1 point in common. Without loss of generality, let $u = w$, so $gh(v) = g(u) = z$ and $gh(u) = g(v) \neq u$, so gh is a translation of step 2, by Lemma 1.5.3.

   If $d_E(x, y) = n \geq 2$, then without loss of generality let $[u, w] \subset [v, z]$, so $d_V(u, w) = n - 1$. Thus $g(u) \notin [u, z]$ and $g(v) \notin [u, g(u)]$, so $u \in [v, gh(v)]$ but $gh(u) \notin [v, g(v)]$ so by Lemma 1.5.3, gh is a translation. Finally, $d_V(v, gh(v)) = d_V(v, u) + d_V(u, w) + d_V(w, z) + d_V(z, g(a)) = 2n$, so the translation is of even step.

ii. Let g be a rotation about v and h an inversion on $\{u, w\}$. See Figure 1.6.

   If $g(u) = u$ but $g(w) \neq w$, then $gh(w) = g(u) = u$ but $gh(u) = g(w) \neq w$ so gh is a translation of step 1.

   If $g(u) \neq u$ and $g(w) \neq w$, then let z be a point such that $g(z) = z$ and that $d_V(z, u)$ is minimum. Without loss of generality, we assume that $d_V(z, w) = d_V(z, u) + 1$. Then $d_V(gh(w), w) = d_V(gh(w), u) + 1 = 2d_V(z, u) + 1$ and $gh(u) = g(w) \notin [gh(w), w] = [g(u), w]$, so again by Lemma 1.5.3, gh is a translation of odd step.

■

1.6 The Automorphism Group of a Tree

Having investigated the properties of the automorphisms, we now turn to look at the properties of the automorphism group of $\Gamma = (V, E)$, which we denote by $\text{Aut}(\Gamma)$. As we proved in Section 1.5, this group is the same as the group of isometries.

We can turn $\text{Aut}(\Gamma)$ into a topological group via the compact open topology:

Proposition 1.6.1 *Given $g \in \text{Aut}(\Gamma)$, and $F \subseteq V$, with F finite, then let*

$$U_F(g) = \{h \in \text{Aut}(\Gamma) : g(x) = h(x), \ \forall\ x \in F\}$$

i. These U_F, together with $\emptyset$ and $\text{Aut}(\Gamma)$ form a basis for a topology τ on $\text{Aut}(\Gamma)$.

ii. The group operations of $\text{Aut}(\Gamma)$ are continuous with respect to τ.

Proof:

i. To show that this is a base, all we need show is that intersections of these sets are some union of elements of the base. Intersections with $\emptyset$ and $\text{Aut}(\Gamma)$ satisfy this condition trivially.

   So given two finite subsets of V, F and H,

$$U_F(f) \cap U_H(h) = \{g \in \text{Aut}(\Gamma) : g(x) = f(x),\ \forall x \in F, \text{ and } g(y) = h(y),\ \forall y \in H\}$$

   If $U_F(f) \cap U_H(h) = \emptyset$, then we are done, so we can assume that there exists some $r \in U_F(f) \cap U_H(h)$. Then

$$\begin{aligned} U_F(f) \cap U_H(h) &= \{g \in \text{Aut}(\Gamma) : g(x) = f(x) = r(x),\ \forall x \in F, \\ &\quad\ \text{and } g(y) = h(y) = r(y),\ \forall y \in H\} \\ &= \{g \in \text{Aut}(\Gamma) : g(x) = r(x),\ \forall x \in F \cup H\} \\ &= U_{F \cup H}(r) \end{aligned}$$

ii. Let (g_λ, f_λ) be a net, $\lambda \in (D, \succeq)$ which converges to (g, f) in $\text{Aut}(\Gamma) \times \text{Aut}(\Gamma)$. That is, given any neighbourhood U of (g, f), there exists some residual $W \subset D$ such that $(g_\lambda, f_\lambda) \in U$ for all $\lambda \in W$.

   Now given any finite subset H of V, we consider the neighbourhood $U_{f^{-1}H}(g) \times U_{f^{-1}H}(f)$, so for all $\lambda \in W$, $(g_\lambda, f_\lambda) \in U_{f^{-1}H}(g) \times U_{f^{-1}H}(f)$. Therefore,

$$\begin{aligned} & f_\lambda(u) &= f(u) & \quad \forall\ u \in f^{-1}H \\ \Rightarrow\ & f_\lambda^{-1}(v) &= f^{-1}(v) & \quad \forall\ v \in H \\ \Rightarrow\ & g_\lambda(f_\lambda^{-1}(v)) &= g_\lambda(f^{-1}(v)), & \quad \forall\ v \in H \\ & &= g(f^{-1}(v)) & \quad \forall\ v \in H \end{aligned}$$

   since $f^{-1}(v) \in f^{-1}H$.

   Thus $g_\lambda f_\lambda^{-1} \in U_H(gf^{-1})$, $\forall \lambda \in W$. Since $U_H(gf^{-1})$ is a neighbourhood base about gf^{-1}, we have that $g_\lambda f_\lambda^{-1}$ converges to gf^{-1}.

   This means that $(g, f) \mapsto gf^{-1}$ is continuous, which gives the continuity of the group operations.

■

It is worthwhile noting here that the U_F are subgroups of $\mathrm{Aut}(\Gamma)$.

This topology has several properties, the most important of which is that it is locally compact.

Definition 1.6.1 *A topological group is* locally compact *if there exists a compact base at the identity.*

Before we can prove local compactness we need the following result:

Theorem 1.6.2 *For all $v \in V$, the stabiliser of v in* $\mathrm{Aut}(\Gamma)$*,*

$$K_v = U_{\{v\}}(e) = \{g \in \mathrm{Aut}(\Gamma) : g(v) = v\}$$

is compact.

Proof:

Every $g \in K_v$ gives a permutation on the set $\partial B_v(n)$, for all $n \geq 1$, since g is an isometry.

So K_v acts as a subgroup of the symmetric group on $B_v(n)$, $S(r_n)$ where $r_n = |B_v(n)|$, so we have a homomorphism $\alpha_n : K_v \to S(r_n)$.

Now we consider the set $\prod_n S(r_n)$. By Tychanoff's theorem, it is compact, since it is a product of finite (and therefore compact) sets. We can map K_v into $\prod_n S(r_n)$ by the map $\alpha : \ g \mapsto (\alpha_1(g), \alpha_2(g), \ldots)$. α is continuous, since $\pi_n \circ \alpha = \alpha_n$, where π_n is the projection from $\prod_n S(r_n)$ onto $S(r_n)$, and the α_n are continuous. This follows from the fact that the topology on $S(r_n)$ is the discrete topology, so we need only worry about the inverse image of some element s of $S(r_n)$. If $s \notin$ the image of K_v then the inverse image is the empty set, so let $g \in K_v$ be such that $\alpha_n(g) = s$. Then $\alpha_n^{-1}(s) = U_{B_v}(n)(g) \cap K_v$, which is open.

Furthermore, α is clearly injective, since if two automorphisms map onto exactly the same element of $\prod_n S(r_n)$, they have exactly the same action on Γ, and so they must be identical.

If g is in the complement of k, then for some n, and for some $v_1, v_2 \in \partial B_v(n)$ we have $g(v_1) = v_2$ but the element there is some $v_3 \in [v, v_1]$ with $d_V(v, v_3) = m$ such that $g(v_3)$ is not the element of $[v, v_2]$ of distance m from v. For a fixed n, the set of such g is clearly open, so the union over all n is also open. Thus the image of K_v is closed and therefore compact and hence K_v is itself compact. ■

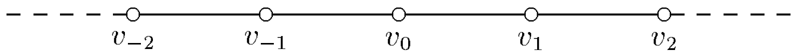
Corollary 1.6.3 $\mathrm{Aut}(\Gamma)$ *is locally compact.*

Proof:

The sets $U_F(e)$ form a neighbourhood base at e. But $U_F(e) = \bigcap_{v \in F} K_v$ is compact, since K_v is compact for all v. ■

This means that $\mathrm{Aut}(\Gamma)$ has a Haar measure defined on it (See section 2.1), and so we may hope to get information about the group's representations from this. We do this in Chapter 2.

Example 1.6.1 *Let Γ be the homogeneous tree of order* 2*, which we will label as follows:*



Clearly $K_{v_0} \cong \mathbf{Z}_2$. *There is only one possible inversion on an edge, and the set of translations is isomorphic to* $\mathbf{Z}$.

By Proposition 1.5.4(i), we can generate all translations of even step and all inversions via two inversions on adjacent edges. Thus we have that this subgroup N *of* $\mathrm{Aut}(\Gamma)$ *is isomorphic to the group*

$$< \{a, b\} \mid a^2 = b^2 = e > = \mathbf{Z}_2 * \mathbf{Z}_2$$

This subgroup is normal, since if g *is an inversion or even step translation, then* hgh^{-1} *is of exactly the same type, ie. is still in this subgroup.*

$K_{v_0} \cap N = e$ *and since the product of a rotation and an inversion on a distinct edge, by Proposition 1.5.4(ii) is a translation of odd step, we get that* $K_{v_0} N = \mathrm{Aut}(\Gamma)$. *Thus* $\mathrm{Aut}(\Gamma)$ *is the internal semidirect product of* K_{v_0} *and* N.

So, $\mathrm{Aut}(\Gamma) \cong (\mathbf{Z}_2 * \mathbf{Z}_2) \rtimes \mathbf{Z}_2$.

In general, the structure is not quite so nice, as we will see in Section 1.8.

1.7 Point Stabiliser Subgroups, K_v

We now investigate some of the properties of K_v.

Proposition 1.7.1 *If* K_v *and* K_v *are a point stabiliser subgroup then*

i. *For all* $g \in Aut(\Gamma)$, $gK_vg^{-1} = K_u$ *where* $u = g(v)$.

ii. K_v *is totally disconnected.*

iii. *If* Γ *is homogeneous of order* $q+1$, *then if* $d_V(v,u) = n$ *then* $K_v \cap K_u$ *has index* $p_v(u) = (q+1)q^{n-1}$ *in* K_v.

Proof:

i. Let $u = g(v)$. If $k \in K_v$ then $k(v) = v$, so

$$gkg^{-1}(u) = gkg^{-1}g(v) = gk(v) = g(v) = u$$

so $gkg^{-1} \in K_u$. Therefore $K_u \subseteq gK_vg^{-1}$.

If $k' \in K_u$ then $k'(u) = u$, so

$$g^{-1}k'g(v) = g^{-1}k'gg^{-1}(u) = g^{-1}k'(u) = g^{-1}(u) = v$$

so $k' \in g^{-1}K_vg$, and thus $K_u = gK_vg^{-1}$.

ii. Let $g, h \in K_v$, with $g \neq h$. Thus there is some vertex u such that $g(u) \neq h(u)$. Now for any $k \notin U_{\{v,u\}}(g)$, $U_{\{v,u\}}(g) \cap U_{\{v,u\}}(f) = \emptyset$, since if $l(u) = k(u)$, $l(u) \neq g(u)$. Thus $U_{\{v,u\}}(g) \cap \bigcup_{f \notin U_{\{v,u\}}(g)} U_{\{v,u\}}(f) = \emptyset$. Furthermore any $l \in K_v$ is either in $U_{\{v,u\}}(g)$ or $\bigcup_{f \notin U_{\{v,u\}}(g)} U_{\{v,u\}}(f)$ so we have K_v as a disjoint union of open sets. Furthermore $h \in \bigcup_{f \notin U_{\{v,u\}}(g)} U_{\{v,u\}}(f)$.

Thus given any two automorphisms, we can find two disjoint open sets which cover all of K_v. Thus K_v is totally disconnected.

iii. For each $w \in \partial B_v(n)$, choose $g_w \in K_v$ such that $g_w(w) = u$. Such a g_w exists because Γ is homogeneous. Then $K_v = \bigcup_{w \in \partial B_v(n)} g_w(K_v \cap K_u)$, and the cosets $g_w(K_v \cap K_u)$ are distinct. Since $|\partial B_v(n)| = (q+1)q^{n+1} = p_v(u)$, we are done.

■

Part (iii) of the proof above can be generalised to any tree where the valence of any vertex u is only a function of $d_V(v,u)$.

We now want to get a better hold on the structure of the point stabiliser subgroups.

Let Γ be homogeneous of order $q+1$. Choose some $o \in V$. Let K_o^r be the restriction of K_o to $B_o(r)$. We note that $K_o^1 \cong S(q+1)$. Then for $r \geq 2$, as sets,

$$K_o^r = K_o^{r-1} \times (\underbrace{S(q) \times \cdots \times S(q)}_{t \text{ times}})$$

where $t = |\partial B_o(r-1)| = (q+1)q^{r-2}$, since any given element of K_o^r when restricted to $B_o(r-1)$ is an element of K_o^{r-1} and each of the $S(q)$ describes the restriction of K_o^r when restricted to the sets $W_u = \{v \in \partial B_o(r) : d_V(v,u) = 1\}$ where $u \in \partial B_o(r-1)$. See Figure 1.7.

So we can clearly embed K_o^{r-1} in K_o^r via the map $\alpha : K_o^{r-1} \to K_o^r$, where $\alpha(k) = (k, (1, \ldots, 1))$, and we can embed $N = S(q) \times \cdots \times S(q)$ in K_o^r via the map $\beta : N \to K_o^r$, where $\beta(s_1, \ldots, s_t) = (1, (s_1, \ldots, s_t))$. Clearly $N \cap K_o^{r-1} = e$ and we can clearly express every element of K_o^r as a product of an element of K_o^{r-1} and an element of N, by the previous paragraph.

Furthermore, N is normal, since we note that N is exactly the vertex stabiliser of $B_o(r-1)$, and $k(s_1, \ldots, s_t)k^{-1}(B_o(r-1)) = kk^{-1}(B_o(r-1)) = B_o(r-1)$.

Hence $K_o^{r-1} = K_o^{r-1} \rtimes N$.

Now given any element of $g \in K_o$ we can describe it uniquely by its action for each r on $B_o(r)$, so we can write $g = (k_1, k_2, \ldots)$, but we note that we must have that the restriction of k_{i+1} to $B_o(i)$ must be the same as the restriction of g to $B_o(i)$.

So K_o can be represented as the subset of $\prod_{r=1}^{\infty} K_o^r$ such that if $g = (k_1, k_2, \ldots) \in K_o$, then $k_{i+1}|_{B_o(i)} = k_i$.

Note that this means that as sets,

$$K_o = S(q+1) \times S(q) \times S(q) \times \cdots$$

which can be used to give more concretely that K_o is compact.

Although the above discussion holds for homogeneous trees, it can be generalised to $K_T = U_T(e)$ for any tree Γ, and for any finite subtree T. In this case the sets which are acted upon by K_T^r are $T_r = \{v \in V : \exists\, u \in T \text{ where } d_V(v,u) \leq r\}$ (see Section 1.3), K_T^1 may be a product of symmetric groups, one for each orbit of K_T on T_1, and the

Figure 1.7: Restrictions of K_o^r

permutation groups also become products of symmetric groups, one for each orbit of K_T on $W_u = \{v \in T_r \setminus T_{r-1} : d_V(v,u) = 1\}$ where $u \in T_{r-1} \setminus T_r$. (Note that here we define $T_{-1} = \partial T$).

1.8 Faithful Transitive Subgroups

We now look at a class of discrete subgroups of $\mathrm{Aut}(\Gamma)$, where Γ is homogeneous of order $q+1$, in an attempt to get a better hold on the structure of $\mathrm{Aut}(\Gamma)$.

Definition 1.8.1 *We say that a subgroup F of* $\mathrm{Aut}(\Gamma)$ *is a* faithful transitive *subgroup if for every $u, v \in V$, there is a $f \in F$ such that $f(u) = v$ (transitivity), and if $F \cap K_v = e$ for all $v \in V$.*

We note that if F is a faithful transitive subgroup of $\mathrm{Aut}(\Gamma)$, then if we choose some vertex o, then the map $f \mapsto fo$ is a bijection, ie. there is a direct correspondence between elements of F and elements of V. Also F is discrete in $\mathrm{Aut}(\Gamma)$, since if $g, f \in F$, $go = fo$, then $g^{-1}fo = o$, thus $g^{-1}f = e$, so $g = f$, which means $U_o(f) \cap F = f$.

If Γ is not homogeneous, then it is impossible for a group to be transitive in this sense, but it is possible to generalise and get a meaningful definition (see Section 1.9).

We can actually get a characterisation of all faithful transitive subgroups of $\mathrm{Aut}(\Gamma)$, but we need the following lemma.

Lemma 1.8.1 *Given a vertex $o \in V$ and a set $A \subset$* $\mathrm{Aut}(\Gamma)$ *such that $|A| = q+1$ and*

i. $A = A^{-1}$

ii. $A(o) = \{a(o) : a \in A\} = \partial B_o(1)$

then

i. given a finite sequence $a_1, \ldots, a_n$ *of elements of* A*, then* $d_V(o, a_1 \ldots a_n(o)) \leq n$*.*

ii. given any vertex $v \in V$*, with* $d_V(v, o) = n$*, it is possible to find a unique finite sequence of* n *elements* $a_1, \ldots, a_n$ *of* A *such that* $a_1 \ldots a_n(o) = v$*, and for this sequence* $a_i a_{i+1} \neq e$*, for* $i = 1, \ldots, n-1$*.*

Proof:

i. We use induction on n.

If $n = 1$, the result is true by assumption.

If the result is true for $n = k$, then given $a_1, \ldots, a_{k+1} \in A$, then

$$d_V(o, a_1 \ldots a_{k+1}(o)) = d_V(a^{-1}(o), a_2 \ldots a_{k+1}(o))$$

since a_1 is an isometry. But

$$\begin{aligned} d_V(a^{-1}(o), a_2 \ldots a_{k+1}(o)) &\leq d_V(a^{-1}, o) + d_V(o, a_2 \ldots a_{k+1}(o)) \\ &\leq 1 + k \end{aligned}$$

So the result is true for $k + 1$.

ii. We note first that since $|A| = q + 1$ and $|\partial B_o(1)| = q + 1$, the second condition on A implies that if $a, b \in A$ and $a(o) = b(o)$, then $a = b$. Thus for each $u \in \partial B_o(1)$, there is a unique $a \in A$ such that $a(o) = u$.

Again, we use induction on n.

If $n = 1$, existance is true by assumption, and uniqueness follows from above.

If the result is true for $n = k$, then given $v \in V$ with $d_V(o, v) = k + 1$, let $[o, v] = v_0, \ldots, v_{k+1}$ be the chain between o and v. Then there is a sequence $a_1, \ldots, a_k$ such that $a_1 \ldots a_k(o) = v_k$, with $a_i a_{i+1} \neq e$, for $i = 1, \ldots, k-1$. Now $d_V(o, (a_1 \ldots a_k)^{-1}(v)) = d_V(v_k, v) = 1$ since each a_i is an isometry. So there is a unique $a \in A$ such that $a(o) = (a_1 \ldots a_k)^{-1}(v)$, therefore $a_1 \ldots a_k a(o) = v$, and furthermore, $a_k a \neq e$, since then $v = a_1 \ldots a_k a(o) = a_1 \ldots a_{k-1}(o)$, and by part (i), $d_V(o, v) = d_V(o, a_1 \ldots a_{k-1}(o)) \leq k - 1$, which is a contradiction.

So the result is true for $k + 1$.

■

Sets of the form of A are easy to find, for instance, a set of inversions, one inverting each edge radiating from o will do. Alternatively, two inversions could be replaced by a step 1 translation and the corresponding step 1 translation in the other direction along the same geodesic.

The above lemma leads to the following classification theorem for faithful transitive subgroups.

Theorem 1.8.2 *Let A be a subset of* Aut(Γ) *as above. Then A generates a faithful transitive subgroup F of* Aut(Γ)*. Furthermore F is isomorphic to the free product of F_s and t copies of $\mathbf{Z}_2$, and these are the only faithful transitive subgroups.*

Proof:

Every element a of A must either be an inversion or a step one translation, by the following argument: If a is a rotation, then $d_V(v, a(v))$ must be even, so $d_V(o, a(o)) \neq 1$. If a is a translation of step $j \geq 2$, then by the proof of Theorem 1.5.2, the minimum distance between any two vertices is j, so $d_V(o, a(o)) \neq 1$. Thus a must either be a step one translation or an inversion. It is an inversion iff $a^2 = e$.

Let $\{b_1, \ldots, b_t\}$ be the subset of A of inversions, and let $\{b_{t+1}, ..., b_{t+s}, b_{t+1}^{-1}, \ldots, b_{t+s}^{-1}\}$ be the step one translations. So elements of the subgroup F of Aut(Γ) generated by A are all words consisting of elements of A where no two adjacent elements are inverses, so

$$F \cong < A \mid b_i^2 = e = b_{t+j}b_{t+j}^{-1}, \ \forall i = 1, \ldots, t, \ j = 1, \ldots, s >$$

so the $b_{t+1}^{-1}, \ldots, b_{t+s}^{-1}$ are clearly redundant, and we can instead write

$$F \cong < \{b_1, \ldots, b_{t+s}\} \mid b_i^2 = e, \ \forall i = 1, \ldots, t >$$

which is exactly the definition of the free product of the free group on s symbols with t copies of $\mathbf{Z}_2$, ie.

$$F \cong F_s * \underbrace{\mathbf{Z}_2 * \cdots * \mathbf{Z}_2}_{t \text{ times}}$$

(See Theorem 1.2.2).

F is clearly transitive by part (ii) of the previous Lemma, and F is faithful by the uniqueness condition from part (ii), since if $fg(v) = v$, then there is some h such that $fgh(o) = h(o)$ by part (ii), and this implies that $gh(o) = f^{-1}h(o)$, which by uniquness means $g = f^{-1}$, so there are no rotations other than e.

Finally, if F is a faithful transitive subgroup, then choose some vertex $o \in V$, and let $A = \{g \in F : g(o) \in \partial B_o(1)\}$, ie. $A(o) = \partial B_o(n)$. Then since F is faithful, $|A| = |\partial B_o(1)| = q + 1$. Furthermore, since $d_V(o, g^{-1}(o)) = d_V(g(o), o) = 1$, then $g^{-1} \in A$, so $A = A^{-1}$, and thus A fits the hypothesis. Therefore A generates a faithful transitive subgroup, and this must be exactly the original subgroup F. ∎

Note that this result means that the faithful transitive subgroups of a homogeneous tree Γ are *exactly* the groups which have Γ as their Cayley graph (see Theorem 1.2.2).

We now hope to use this subgroup along with the point stabilisers, to get additional information about the structure of Aut(Γ). We might hope that since $F \cap K_v = e$, and by Example 1.6.1 that we might be able to form the semidirect product of the two subgroups and get the whole automorphism group. Unfortunately, F is not, in general, normal. However, we do have the following.

Proposition 1.8.3 *$Aut(\Gamma) = K_oF = FK_o$ and furthermore if $g = f_1k_1 = k_2f_2$, $k_1, k_2 \in K_o$, $f_1, f_2 \in F$, then k_1, f_1, k_2, f_2 are unique.*

Proof:

Let $g \in$ Aut(Γ). Then there exists a unique $f_1 \in F$ such that $f_1(o) = g(o)$, so there is a unique $k_1 \in K_o$ such that $k_1 = f_1^{-1}g$, since $f_1^{-1}g(o) = o$, and hence $g = f_1k_1$.

Similarly there is some vertex v such that $g(v) = o$, so there is a unique $f_2 \in F$ such that $f_2(v) = o$. So there is a unique $k_2 = gf_2^{-1} \in K_o$, since $gf_2^{-1}(o) = g(v) = o$, and hence $g = k_2 f_2$ ■

This means that, as sets, $\mathrm{Aut}(\Gamma) = F \times K_o = K_o \times F$, which allows is to prove that the Haar measure on $\mathrm{Aut}(\Gamma)$ is unimodular (see section 2.1).

1.9 Extensions and Generalisations

In this section we deal with some related topics to those we have already discussed.

Unimodular Subgroups of $\mathrm{Aut}(\Gamma)$

Although we have not yet defined unimodularity, now is an appropriate time to discuss what conditions might lead to unimodularity. If the reader is unfamiliar with the concept of unimodularity, then the appropriate definitions may be found in Section 2.1.

As we will see, if we want to perform harmonic analysis on a group which acts on a tree, it is critical that the above Proposition holds, since it is by this that we will prove unimodularity.

To retain the subgroup as being locally compact it has to be a closed subgroup, so only those which are closed need to be considered.

If we want unimodularity, by the method given here, then we must be able to write the subgroup G of $\mathrm{Aut}(\Gamma)$ as a product of a compact subgroup K of $\mathrm{Aut}(\Gamma)$ and a faithful transitive subgroup F.

However, this is not the only way of showing unimodularity. Figà-Talamanca [4] uses the idea of Gelfand pairs to show unimodularity. It turns out that it is sufficient to show that a group is both transitive on the vertices of Γ and on its boundary Ω to show that the group is unimodular.

Clearly these sorts of conditions fall far short of being neccessary, since any compact subgroup of $\mathrm{Aut}(\Gamma)$ is unimodular simply by virtue of being compact. In particular the point stabiliser subgroups K_v, and more generally K_T are unimodular, but they are clearly neither contain faithful transitive subgroups, nor are they transitive on vertices and even in the case of K_T neccesarily, they are not neccesarily even transitive on Ω.

The Explicit Structure of a Group Acting on Tree

It is possible to get hold of the structure of a general group acting on a tree. These results come from Serre [11].

We begin with the following definition.

Definition 1.9.1 *Given a tree $\Gamma = (V, E)$, and a subgroup G of $\mathrm{Aut}(\Gamma)$, then we define*

i. The quotient graph *$\Gamma/G = (U, D)$, where U is the set of orbits of G in V and D is the set of orbits of G in E, and an element a of U is the endpoint of an element b in D if there is some vertex v in a such that it is the endpoint of some y in b.*

ii. A tree of representatives *of the quotient graph is a subtree T of Γ such that T and Γ/G are isomorphic.*

Example 1.9.1 *If Γ is the semihomogeneous tree of order $(q+1, r+1)$, then V has two orbits under the action of* $\mathrm{Aut}(\Gamma)$*: the vertices of order $q+1$, and the vertices of order $r+1$, and E has one orbit. Thus the quotient graph is a single edge and its two endpoints.*

Therefore any edge of Γ and its endpoints is a tree of representatives.

Note that there is no reason why a tree of representatives must exist, however if G contains no inversions, then Γ/G is a tree, and there must be a tree of representatives (see Serre [11, §3.1, p25]).

The fact the G must contain no inversions, as it turns out, is not a major problem, since if we take the barycentric subdivision of Γ, G has a well defined action on Γ' which is inversion free. We map each $g \in G$ to $g' \in \mathrm{Aut}(\Gamma')$, where $g'(v) = g(v)$, for $v \in V$ and $g'(y) = g'(\{u, v\}) = \{g(u), g(v)\}$, for $y = \{u, v\} \in E$. The fact this is inverse free is clear since if g is an inversion on Γ, then g stabilises some edge y, and so g' is a rotation about the corresponding vertex of Γ'. (In fact it can be shown, that if Γ is not the homogeneous tree of order 2, then $\mathrm{Aut}(\Gamma) \cong \mathrm{Aut}(\Gamma')$).

Serre shows that it is possible to derive the structure of many groups from their actions on a tree, however the simplest example is the case when the tree of representatives is an edge $y = \{u, v\}$, then $G \cong K_u *_{K_y} K_v$, the amalgamated product of the point stabilisers of u and v amalgamated over the point stabiliser of y, that is the $\{g \in G : gv = v,\ gu = u\}$. There are more general results, however they are outside the scope of this thesis.

Chapter 2

Representations of Groups on a Tree

In this chapter we develop some initial theory concerning harmonic analysis and representations, before setting out to classify and identify the irreducible representations of $\mathrm{Aut}(\Gamma)$. After the initial sections, we essentially follow chapter III of Figà-Talamanca [4].

2.1 Haar Measures

Most of the following section, except that dealing specifically with groups acting on trees, can be found in any good book on Harmonic Analysis. In particular, the proofs of the general theorems can be found in Halmos [6].

We begin by producing a particularly nice measure on a topological group.

Theorem 2.1.1 (Haar, Von Neumann) *Given a locally compact group G, with Borel sets $\mathcal{B}$ there exists a Borel measure μ on G, with the following properties:*

i. If $V \neq \emptyset$ is open, then $\mu(V) > 0$.

ii. There exists an open set V with $\mu(V) < \infty$.

iii. If $X \in \mathcal{B}$, then

$$\mu(gX) = \mu(X), \ \forall \ g \in G.$$

(ie. μ is left invariant*).*

iv. μ is regular*, that is $\mu(E) = \inf\{\mu(V) : E \subset V, \ V \text{ open}\}$ and $\mu(E) = \sup\{\mu(C) : C \subset E, \ C \text{ compact}\}$*

Such a measure is called a left Haar measure *and is unique up to a multiplicative constant, that is if μ and ν are both left Haar measures, then for some $c \in R_+$, $\mu(X) = c\nu(X), \ \forall \ X \in \mathcal{B}$*

Similarly, there exists a right Haar measure *on G.*

(Existence is due to Haar, and uniqueness due to Von Neumann).

Proof: See Halmos [6, XI; §58,60].

Example 2.1.1 *The following are some examples of Haar measures.*

i. On $R, +$ the Lebesgue measure is both a left and right Haar measure.

ii. On $Z, +$ the counting measure is both a left and right Haar measure.

iii. If G is compact, then $\mu(G) < \infty$, so we can normalise the measure so that $\mu(G) = 1$.

iv. Similarly, if G is discrete, then we can normalise the measure by setting $\mu(\{e\}) = 1$, making μ the counting measure on G.

v. Given two locally compact groups G_1 and G_2, with Haar measures μ_1 and μ_2, then we can define $\mu_1 \times \mu_2$ on $G_1 \times G_2$. This measure is a Haar measure on $G_1 \times G_2$.

vi. If G has finite Haar measure and H is a closed normal subgroup, then G/H is a locally compact group and has Haar measure $\mu \circ \theta^{-1}$, where $\theta : G \to G/H$ is the quotient map.

vii. If H is a subgroup of G, such that $\mu(H) > 0$, then the restriction of μ to H is a Haar measure on H.

A natural question arises: When is the Haar measure both left and right invariant? That is, when is it *bi-invariant*? If this occurs, we call the group *unimodular*.

Proposition 2.1.2 *The following groups are unimodular:*

i. All locally compact Abelian groups.

ii. All compact groups.

iii. All discrete groups.

Proof: See Rieter [9, III, §3.5-6, p62].

Proposition 2.1.3 $\mathrm{Aut}(\Gamma)$ *is unimodular.*

Proof:

Since each element of $\mathrm{Aut}(\Gamma)$ can be expressed uniquely as a product of an element of K_v and an element of F, a faithful transitive subgroup, $\mathrm{Aut}(\Gamma) \cong K_v \times F$ as topological spaces. Let μ be right-invariant Haar measure on $\mathrm{Aut}(\Gamma)$. Choose some $v \in V$, since K_v is compact $\mu(K_v) < \infty$, so we shall normalise μ so that $\mu(K_v) = 1$.

Then $\mu_u = \mu$ restricted to K_u is a Haar measure on K_u, since K_u is an open subgroup of $\mathrm{Aut}(\Gamma)$. Since $K_u \cap K_v$ has index $(q+1)q^{n-1}$ in K_v where $d_V(u, v) = n$, $\mu(K_v \cap K_u) = \mu_v(K_v \cap K_u) = \frac{1}{(q+1)q^{n-1}}$

Furthermore, the measures on each of the K_v are essentially identical, since for $E \subseteq K_{fv}$, $\mu_{fv}(E) = \mu_v(f^{-1}Ef)$, as follows. $\mu_v(f^{-1}.f)$ is certainly a measure on K_{fv}. For all $k \in K_{fv}$,

$$\begin{aligned} \mu_v(f^{-1}kEf) &= \mu_v(f^{-1}kff^{-1}Ef) \\ &= \mu_v(f^{-1}Ef) \end{aligned}$$

since $f^{-1}kf \in K_v$ and μ_v is a bi-invariant Haar measure. Thus $\mu_v(f^{-1}.f)$ is a Haar measure, so $\mu_v(f^{-1}Ef) = c\mu_{fv}(E)$. But

$$\begin{aligned}
\mu_v(f^{-1}(K_v \cap K_{fv})f) &= \mu_v(K_{f^{-1}v} \cap K_v) \\
&= \frac{1}{(q+1)q^{n-1}} \\
\mu_{fv}(K_v \cap K_{fv}) &= \mu(K_v \cap K_{fv}) \\
&= \mu_v(K_v \cap K_{fv}) \\
&= \frac{1}{(q+1)q^{n-1}}
\end{aligned}$$

Since if $d_V(v, fv) = n$, then $d_V(v, f^{-1}v) = n$ also. Therefore $c = 1$, and we have our result.

Given $E \in \mathrm{Aut}(\Gamma)$, E Borel, then we define for each $u \in V$, $f \in F$, $E_f(u) = (E \cap K_u f)f^{-1} \subseteq K_u$. Intuitively, this is "slicing" E into sections, one for each coset of K_u, and then pulling these back to K_u. Therefore E is the disjoint union of $E_f(u)f$ as f varies over F, ie. $E = \bigcup_{f \in F} E_f(u)f$. Now since F is countable, we have

$$\begin{aligned}
\mu(E) &= \sum_{f \in F} \mu(E_f f) \\
&= \sum_{f \in F} \mu(E_f) \\
&= \sum_{f \in F} \mu_u(E_f)
\end{aligned}$$

for any $u \in V$.

Now we are in a position to prove unimodularity. Let $E \subseteq \mathrm{Aut}(\Gamma)$, E Borel.

Let $k \in K_v$, then

$$\begin{aligned}
\mu(kE) &= \sum_{f \in F} \mu_v((kE)_f(v)) \\
&= \sum_{f \in F} \mu_v((kE \cap K_v f)f^{-1}) \\
&= \sum_{f \in F} \mu_v(k(E \cap k^{-1}K_v f)f^{-1}) \\
&= \sum_{f \in F} \mu_v(k(E \cap K_v f)f^{-1}) \\
&= \sum_{f \in F} \mu_v((E \cap K_v f)f^{-1}) \\
&= \sum_{f \in F} \mu_v(E_f(v)) \\
&= \mu(E)
\end{aligned}$$

Let $h \in F$, then

$$\mu(hE) = \sum_{f \in F} \mu_v((hE)_f(v))$$

$$
\begin{aligned}
&= \sum_{f\in F} \mu_v((hE\cap K_vf)f^{-1}) \\
&= \sum_{f\in F} \mu_v(h(E\cap h^{-1}K_vf)f^{-1}) \\
&= \sum_{f\in F} \mu_{h^{-1}v}(h^{-1}h(E\cap h^{-1}K_vf)f^{-1}h) \\
&= \sum_{f\in F} \mu_{h^{-1}v}((E\cap h^{-1}K_vhh^{-1}f)f^{-1}h) \\
&= \sum_{f\in F} \mu_{h^{-1}v}((E\cap K_{h^{-1}v}h^{-1}f)f^{-1}h) \\
&= \sum_{g\in F} \mu_{h^{-1}v}((E\cap K_{h^{-1}v}g)g^{-1}) \\
&= \sum_{g\in F} \mu_{h^{-1}v}((E_g(h^{-1}v)) \\
&= \mu(E)
\end{aligned}
$$

And since any element of Aut(Γ) can be expressed as a product of elements from these two sets, we are done. ■

From this proof we know that $\mu(K_v) = \mu(K_u) < \infty$ for all $u, v \in V$, so we can normalise the Haar measure in such a way that $\mu(K_v) = 1$, for all $v \in V$. We will assume that this normalisation has taken place.

2.2 Representations

In this section we explore some general ideas from representation theory.

Definition 2.2.1 *Given a group G,*

A representation π *of G is a homomorphism from G to the group of linear operators on some Hilbert Space $\mathcal{H}_\pi$, ie. $\forall\ g, h \in G$ we have*

$$
\begin{aligned}
\pi(g)\pi(h) &= \pi(gh) \\
\pi(g^{-1}) &= \pi(g)^{-1}
\end{aligned}
$$

In addition, we require that $\forall\ \xi, \eta \in \mathcal{H}_\pi$, $\langle \pi(g)\xi, \eta\rangle$ is continuous as a function of g.

A representation is unitary *if $\pi(g)$ is a unitary operator for all $g \in G$. (Recall that an operator U on $\mathcal{H}_\pi$ is unitary if $U^* = U^{-1}$, where U^* is the operator such that $\langle U\xi, \eta\rangle = \langle \xi, U^*\eta\rangle$, $\forall\ \xi, \eta \in \mathcal{H}_\pi$).*

A representation is irreducible *if there is no nontrivial closed subspace of $\mathcal{H}_\pi$ invariant under the action of π. ie. the closed span of $\{\pi(g)\xi : g \in G\}$ is $\mathcal{H}_\pi$, $\forall\ \xi \in \mathcal{H}_\pi$.*

A representation is faithful *if it is injective.*

Two representations π and ϕ are (unitarily) equivalent *if there exists a surjective linear isometry $U : \mathcal{H}_\pi \to \mathcal{H}_\phi$ such that $U\pi(g) = \phi(g)U$ for all $g \in G$.*

Given two representations π and ϕ,we define their direct sum $\pi \oplus \rho$ *to be the representation from $G \to \mathcal{H}_\pi \oplus \mathcal{H}_\phi$ defined by $(\pi \oplus \rho)(g) = (\pi(g), \rho(g))$.*

For the rest of this section we will assume that G is unimodular and that all representations will unitary.

There are two representations which should be introduced immediately.

Definition 2.2.2 *The* left regular representation λ *on a locally compact group* G*, with Haar measure* μ*, is a representation on* $\mathcal{H}_\lambda = L^2(G,\mu)$ *defined by*

$$\lambda(g)f(k) = f(g^{-1}k)$$

where $g, k \in G$*,* $f \in L^2(G)$*.*

The right regular representation ρ *is defined similarly, but with*

$$\rho(g)f(k) = f(kg)$$

There is also a class of representations that we will need to know more about. This section comes from Dixmier [2].

Definition 2.2.3 *A representation* π *is* square integrable *if* $\exists \xi \in \mathcal{H}_\pi$*, with* $\xi \neq 0$*, such that* $g \mapsto \langle \pi(g)\xi, \xi \rangle$ *is in* $L^2(G)$*.*

Proposition 2.2.1 *If* π *is an irreducible square integrable representation, then*

i. The coordinate functions $t_{\xi,\nu} : g \mapsto \langle \pi(g)\xi, \nu \rangle$ *are* L^2 *for all* $\xi, \nu \in \mathcal{H}_\pi$*.*

ii. There exists a number d_π*,* $0 < d_\pi < \infty$*, such that*

$$\int_G t_{\xi,\nu}(g)\overline{t_{\xi',\nu'}(g)}\ d\mu(g) = \frac{\langle \xi, \xi' \rangle \langle \nu, \nu' \rangle}{d_\pi}$$

We call d_π *the* formal dimension *of* π*.*

Proof: See Dixmier [2, p278–282].

Now if we fix ν_0, where $\|\nu_0\|^2 = 1$ and set $(U\xi)(g) = \sqrt{d_\pi}\, t_{\xi,\nu_0}(g)$, then from the above propsition,

i. $U\xi \in L^2$

ii. U is an isometry from $\mathcal{H}_\pi$ to L^2, since

$$\begin{aligned}\langle U\xi, U\xi' \rangle &= d_\pi \frac{\langle \xi, \xi' \rangle \langle \nu_0, \nu_0 \rangle}{d_\pi} \\ &= \langle \xi, \xi' \rangle\end{aligned}$$

iii. U *intertwines* π and ρ as follows:

$$\begin{aligned}(U\pi(g)\xi)(h) &= \sqrt{d_\pi}\langle \pi(h)\pi(g)\xi, \nu_0 \rangle \\ &= (U\xi)(hg) \\ &= \rho(g)(U\xi)(h)\end{aligned}$$

This means that π is unitarily equivalent to some subrepresentation of ρ.

It is also possible to define a function $s_{\xi,\nu} : g \mapsto \langle \xi, \pi(g)\nu \rangle$, in which case corresponding results give that π and λ can be intertwined, and that π is also unitarily equivalent to a subrepresentation of λ.

This means that given any irreducible square integrable representation π on $\mathcal{H}_\pi$ we can consider $\mathcal{H}_\pi$ as a subspace of L^2 and π as the left (or right) regular representation restricted to this subset.

We also need the concept of induced representations. The following comes from Figà-Talamanca [4, III, §3, p132]. It concerns only the special case of induced representations of unimodular, separable, locally compact groups and compact open subgroups.

Let G be a unimodular, separable, locally compact group, and K a compact open subgroup of G. Let σ be a unitary representation on K.

Definition 2.2.4 *Let S^σ be the space of functions $f : G \to \mathcal{H}_\sigma$ such that*

i. $f(gh) = \sigma(h^{-1})f(g)$ *for ever* $g \in G$ *and* $h \in K$.

ii. $\int_G \|f(g)\|^2 \, dg < \infty$

This gives us that

$$\int_G \|f(g)\|^2 \, dg = \mu(K) \sum_{x \in G/K} \|f(g)\|^2$$

and S^σ is a Hilbert space with inner product

$$\langle f, g \rangle = \int_G \langle f(x), g(x) \rangle \, dx$$

S^σ is invariant under left translation, so we can define a left regular representation of G on S^σ, which we denote $\mathrm{Ind}(\sigma)$, that is

Definition 2.2.5 *We define the* induced representation *of σ to be*

$$Ind(\sigma)(x)f(t) = f(x^{-1}t)$$

Two representations σ_1 and σ_2 are unitarily equivalent, iff their induced representations are also unitarily equivalent.

Also given two representations σ_1 and σ_2, we have that $\mathrm{Ind}(\sigma_1 \oplus \sigma_2)$ is unitarily equivalent to $\mathrm{Ind}(\sigma_1) \oplus \mathrm{Ind}(\sigma_2)$.

The if σ is square integrable, so is $\mathrm{Ind}\sigma$, and the formal dimension of $\mathrm{Ind}(\sigma)$ is $dim(\sigma)/\mu(K)$.

If S^σ_K is the subspace of S^σ of functions supported on K, then S^σ_K is a closed non-trivial subspace of S^σ. If we define for $\xi \in \mathcal{H}_\sigma$ the functions f_ξ where $f_\xi(x) = \sigma(x^{-1})\xi$ for $x \in K$ and $f_\xi(x) = 0$ otherwise, then the map $\xi \mapsto f_\xi$ is an isomorphism of $\mathcal{H}_\sigma$ onto S^σ_K.

Thus any function in S^σ with compact support is a linear combination of left translates of functions in S^σ_K, and the subspace of S^σ generated by $\bigcup_{g \in G} \lambda(g) S^\sigma_K$ is dense in S^σ.

Proposition 2.2.2 *If σ is a unitary irreducible representation of K, then $Ind(\sigma)$ is irreducible iff every closed non-trivial invariant subspace of S^σ contains a non-trivial function of S^σ_K.*

Proof: See Theorem 3.13 of Figà-Talamanca [4, III, §3, p134].

2.3 Representations of $\text{Aut}(\Gamma)$

We now seek to classify all irreducible representations of $\text{Aut}(\Gamma)$, and develop some tools to help deal with them.

If K is a compact subgroup of $\text{Aut}(\Gamma)$ with normalised Haar measure μ_K, then given a unitary representation π on $\mathcal{H}_\pi$, we fix some $\xi \in \mathcal{H}_\pi$, and define a map

$$\eta \mapsto \int_K \langle \pi(k)\xi, \eta \rangle \; d\mu_K(k)$$

This map is bounded conjugate linear map, so by the Riesz representation theorem, there exists some $\xi' \in \mathcal{H}_\pi$ such that this map is

$$\nu \mapsto \langle \xi', \nu \rangle$$

ξ' depends on ξ in a bounded linear fashion, and hence ξ' is of the form $P_\pi(K)\xi$ for some linear operator $P_\pi(K)$. Furthermore, $P_\pi(K)$ is bounded, since

$$\begin{aligned} |\langle P_\pi(K)\xi, \eta \rangle| &\leq \int_K |\langle \pi(K)\xi, \eta \rangle| \; d\mu_K(k) \\ &\leq \|\xi\|.\|\eta\| \end{aligned}$$

So $\|P_\pi(K)\| \leq 1$, and clearly now

$$\langle P_\pi(K)\xi, \eta \rangle = \int_K \langle \pi(K)\xi, \eta \rangle \; d\mu_K(k)$$

This leads to the following proposition:

Proposition 2.3.1 *For such a $P_\pi(K)$ we have:*

i. Given $k \in K$, we have $\pi(k)P_\pi(K) = P_\pi(K)$.

ii. If $\pi(k)\xi = \xi$, $\forall\ k \in K$, then $P_\pi(K)\xi = \xi$.

iii. $P_\pi(K)$ is the orthogonal projection onto the space of $\pi(k)$-invariant vectors.

Proof:

i. For all $\xi, \eta \in \mathcal{H}_\pi$,

$$\begin{aligned} \langle \pi(g)P_\pi(K)\xi, \eta \rangle &= \langle P_\pi(K)\xi, \pi(g)^*\eta \rangle \\ &= \int_K \langle \pi(k)\xi, \pi(g)^*\eta \rangle dk \\ &= \int_K \langle \pi(g)\pi(k)\xi, \eta \rangle dk \\ &= \int_K \langle \pi(gk)\xi, \eta \rangle dk \\ &= \int_K \langle \pi(k')\xi, \eta \rangle dk' \\ &= \langle P_\pi(K)\xi, \eta \rangle \end{aligned}$$

So $\pi(k)P_\pi(K) = P_\pi(K)$.

ii. For all $\eta \in \mathcal{H}_\pi$,

$$\begin{aligned}
\langle P_\pi(K)\xi, \eta\rangle &= \int_K \langle \pi(k)\xi, \eta\rangle dk \\
&= \int_K \langle \xi, \eta\rangle dk \\
&= \langle \xi, \eta\rangle \int_K dk \\
&= \langle \xi, \eta\rangle
\end{aligned}$$

So $P_\pi(K)\xi = \xi$.

iii. $P_\pi(K)$ is an orthogonal projection if $P_\pi(K)P_\pi(K) = P_\pi(K)$ and $P_\pi(K)^* = P_\pi(K)$. This follows from (i) and (ii), since

$$\begin{aligned}
\langle P_\pi(K)P_\pi(K)\xi, \eta\rangle &= \int_K \langle \pi(k)P_\pi(K)\xi, \eta\rangle dk \\
&= \int_K \langle P_\pi(K)\xi, \eta\rangle dk \\
&= \langle P_\pi(K)\xi, \eta\rangle
\end{aligned}$$

and

$$\begin{aligned}
\langle P_\pi(K)^*\xi, \eta\rangle &= \langle \xi, P_\pi(K)\eta\rangle \\
&= \int_K \langle \xi, \pi(k)\eta\rangle dk \\
&= \int_K \langle \pi(k)^*\xi, \eta\rangle dk \\
&= \int_K \langle \pi(k)^{-1}\xi, \eta\rangle dk \\
&= \int_K \langle \pi(k^{-1})\xi, \eta\rangle dk \\
&= \langle P_\pi(K)\xi, \eta\rangle
\end{aligned}$$

as required.

■

If T is a finite subtree of Γ, then let K_T be the subgroup of G that fixes the points of T, and let $P_\pi(T) = P_\pi(K_T)$. P is well-defined, since K_T is the intersection of the stabilisers of each of the vertices of T, and is thus the intersection of a finite number of compact subgroups, and is so therefore itself compact. Similarly, since each of the point stabiliser subgroups is open, K_T is open. In particular $\mu_G(K_T) > 0$, so by the properties of Haar measures,

$$P_\pi(T)\xi = \frac{1}{\mu_G(K_T)} \int_{K_T} \pi(k)\xi \; d\mu_G(k)$$

Also, as T varies over all finite complete subtrees, the K_T form an open neighbourhood basis of e. This gives us that for all $\xi \in \mathcal{H}_\pi$, $\xi \neq 0$, there is a finite complete subtree T such that

$$\int_{K_T} \langle \pi(k)\xi, \xi\rangle \; d\mu_{K_T}(k) \neq 0$$

This follows since $\langle \pi(g)\xi, \xi \rangle$ is continuous in g, and

$$\langle \pi(e)\xi, \xi \rangle = \langle \xi, \xi \rangle = \|\xi\|^2 \neq 0$$

giving a real number ε such that $0 < \varepsilon < \|\xi\|^2$, so there is a finite complete subtree T for which $\pi(K_T)\xi \subseteq \{\eta \in \mathbf{C} \ : \ |\eta - \|\xi\|^2| < \varepsilon\}$, so $\mathrm{Re}(\pi(g)\xi) > 0$ for all $g \in K_T$, so

$$\mathrm{Re}(\int_{K_T} \langle \pi(k)\xi, \xi \rangle \ d\mu_{K_T}(k)) > 0$$

giving the result.

A direct consequence of this is that for each $\xi \neq 0$, there is a tree T such that $P_\pi(T)\xi \neq 0$.

Thus we can make the following definitions:

Definition 2.3.1 *Let π be an irreducible unitary representation of G. Then there is some largest positive integer ℓ_π such that if T is a finite complete subtree for which $P_\pi(T) \neq 0$ then $\ell_\pi \leq$ the number of vertices of T.*

If $\ell_\pi = 1$ then π is called spherical.

If $\ell_\pi = 2$ then π is called special.

If $\ell_\pi > 2$ then π is called cuspidal.

A finite complete subtree T with ℓ_π vertices, for which $P_\pi(T) \neq 0$ is called a minimal tree *of π.*

We note that if T is a minimal tree, then gT is also a minimal tree for all $g \in G$, since $P_\pi(gT) = \pi(g)P_\pi(T)\pi(g)^{-1} \neq 0$, and the number of vertices is clearly the same.

Definition 2.3.2 *If T is any complete finite subtree, then we define*

i. $\mathcal{H}_\pi(T)$ to be the space of all K_T-invariant vectors, ie. $\mathcal{H}_\pi(T)$ is the range of $P_\pi(T)$.

ii. V_π to be the union over all finite complete subtrees T of the $\mathcal{H}_\pi(T)$.

It is easy to show (Figà-Talamanca [4, III, §1, p106]) that V_π is dense in $\mathcal{H}_\pi$, even if π is not irreducible.

Let π be a unitary representation, and let T be a subtree of π with ℓ_π vertices. Thus if T° is a proper complete subtree of T, then $P_\pi(T^\circ)\xi = 0$, for every $\xi \in \mathcal{H}_\pi$. Thus if $\xi, \eta \in V_\pi$ and $P_\pi\xi = \xi$, then the coefficient function $u(g) = \langle \pi(g)\xi, \eta \rangle$ has the following properties:

i. $u(gk) = \langle \pi(g)\pi(k)\xi, \eta \rangle$ for all $k \in K_T$.

ii. There is a finite complete subtree T' such that $u(kg) = \langle \pi(g)\xi, \pi(k^{-1})\eta \rangle$ for all $k \in K_{T'}$. (T' is any subtree for which $P_\pi(T')\eta = \eta$).

iii. If T° is a proper complete subtree of T, then

$$\frac{1}{\mu_G(K_{T^\circ})} \int_{K_{T^\circ}} u(gk) \ d\mu_G(k) = \langle P_\pi(T^\circ)\xi, \pi(g^{-1})\eta \rangle = 0$$

for all $g \in G$.

To study the coefficient functions more closely, we generalise the notion as follows:

Definition 2.3.3 *Let T be a complete finite subtree of Γ. We let $\mathcal{S}(T)$ be the linear space of continuous functions u satisfying the following properties:*

i. u is right-K_T-invariant.

ii. u is left-K_{T_u}-invariant for some finite complete subtree T_u, dependant on u.

iii. for every proper complete subtree $T^\circ \subset T$

$$\int_{K_{T^\circ}} u(gk)\ d\mu_G(k) = 0$$

for all $g \in G$.

$\mathcal{S}(T)$ is G-left-invariant since, if $u \in \mathcal{S}(T)$, then for any $g \in G$, $v(h) = u(gh) \in \mathcal{S}(T)$, as

i. For all $k \in K_T$, we have $v(hk) = u(ghk) = u(gh) = v(h)$, since u is right-K_T-invariant.

ii. Given $k \in K_{g^{-1}T'_u}$, we have $v(kh) = u(gkh) = u(gg^{-1}fgh) = u(fgh) = u(gh) = v(h)$, where $f \in K_{T'_u}$. Thus v is left-$K_{g^{-1}T'_u}$-invariant.

iii. Given $T^\circ \subset T$, $h \in \mathrm{Aut}(\Gamma)$,

$$\begin{aligned} \int_{K_{T^\circ}} v(hk)\ d\mu_G(k) &= \int_{K_{T^\circ}} v(hk)\ d\mu_G(k) \\ &= \int_{K_{T^\circ}} u(ghk)\ d\mu_G(k) \\ &= 0 \end{aligned}$$

Thus if we could show that $\mathcal{S}(T) \subseteq L^2(\mathrm{Aut}(\Gamma))$, $\mathcal{S}(T)$ would be an invariant subspace for λ.

It is easy to see, using similar arguments to above, that under right multiplication by g, $\mathcal{S}(T)$ becomes $\mathcal{S}(gT)$.

Now for a spherical representation π, if we consider $\mathcal{S}(v)$, where v is the minimal tree for π, then condition (iii) becomes trivial, so it is difficult to use $\mathcal{S}(T)$ to extract any information about π.

2.4 Special Representations

Let π be a special representation. Therefore all minimal trees of π must be edges. Let $y = \{a, b\} \in E$ be an edge which is minimal for π.

Proposition 2.4.1 *If π is a special representation, then all edges $y' = \{a', b'\} \in E$ are minimal.*

Figure 2.1: The Orientation E_+

Proof:

If y is a minimal edge, then there is a $g \in \mathrm{Aut}(\Gamma)$ such that $y' = gy$. Thus $P_\pi(y') = \pi(g)P_\pi(y)\pi(g^{-1}) \neq 0$. ■

Consider $\mathcal{S}(y)$. We can replace, for some n_u, T_u by $B_y(n_u)$ which is the tree with vertices $\{v \in V : d_V(v, w) \leq n_u \text{ for some } w \in y\}$ since for n_u sufficiently large, $T_u \subseteq B_y(n_u)$, so $K_{B_y(n_u)} \subseteq K_{T_u}$, giving the result. We will call this property (1).

Also remembering that $\mathcal{S}(y)$ is left invariant, we have, for all $u \in \mathcal{S}(y)$ and for all $h, g \in \mathrm{Aut}(\Gamma)$,

$$\int_{K_a} u(h^{-1}kg)\, dk = 0$$

so, if we set $g = hf$, then

$$\begin{aligned} 0 &= \int_{K_a} u(h^{-1}khf)\, dk \\ &= \int_{h^{-1}K_a h} u(kf)\, dk \\ &= \int_{K_{ha}} u(kf)\, dk \end{aligned}$$

We will call this property (2).

Now consider the following orientation of Γ. We define E_+ to be the set of directed edges such that a is a local source and b is a local sink, and the pattern repeats throughout Γ. This is best described by illustration: see Figure 2.1. We define $E_- = \overline{E_+}$.

Since $\text{Aut}(\Gamma)$ acts transitively on directed edges, and K_y stabilises $[a, b]$, then we can identify G/K_y with $E_+ \cup E_-$ via the map $\alpha : g \mapsto [ga, gb]$. So we define $\tilde{\mathcal{S}}$ to be the space of functions defined by $\tilde{u} \in \tilde{\mathcal{S}}$ iff $u(g) = \tilde{u}(\alpha(g))$.

Since any edge $y' \in E$ has two directed edges associated with it, we define $u_+(y')$ to be the value of $\tilde{u}$ on the positively oriented component of y', and $u_-(y')$to be the value of $\tilde{u}$ on the negatively oriented component of y'.

We can now transfer the definition of $\mathcal{S}(y)$ across to $\tilde{\mathcal{S}}$, and get the following Lemma, however first we define $\mathcal{N}(a, x) = \mathcal{N}_\omega(a, x)$ where ω is any infinite chain which starts with $[a, x]$.

Lemma 2.4.2 *A function $\tilde{u}$ on $E_+ \cup E_-$ is in $\tilde{\mathcal{S}}$ iff*

i. There exists an $n > 0$ such that $\forall\ x$ such that $d_V(a, x) = n$, then $u_+(y')$ and $u_-(y')$ depend only on the distance of y' from a, $d_C(\{a\}, y')$, for $y' \in \mathcal{N}(a, x)$.

ii. For all $v \in V$, and $m \in \mathbf{N}$,

$$\sum_{d_C(\{v\},y')=m} u_+(y') = 0$$

and

$$\sum_{d_C(\{v\},y')=m} u_+(y') = 0$$

Proof:

These two properties are essentially the translation of conditions (1) and (2) to $\tilde{\mathcal{S}}$.

i. Property (1) gives us that u is left-$K_{B_y(n_u)}$-invariant, and so $\tilde{u}$ must be constant on the orbits of $K_{B_y(n_u)}$ on $E_+ \cup E_-$. However, the orbits of $K_{B_y(n_u)}$ are exactly the edges in $\mathcal{N}(a, x)$ of a given distance and orientation from $x \in \partial B_y(n_u)$, so they must all have the same value. Thus the two conditions are equivalent.

ii. Property (2) gives, given some vertex v,

$$\int_{K_v} u(kf)\ dk = 0$$

which translated to $E_+ \cup E_-$ gives that the sum of $\tilde{u}$ within an orbit of K_v must be 0. The edges of a given distance and orientation from v are exactly these orbits. So again the two conditions are equivalent.

Thus the lemma follows. ■

From this we can see that each element of $\tilde{\mathcal{S}}$ is determined by its values on some finite set of edges, since all the edges outside $B_y(n_u)$ are determined by the values on $\partial B_y(n_u)$.

More specifically, if $y' \in \mathcal{N}(a, x)$ and $u_+(\{x, t\}) = c$ where $[a, x] = a, \ldots, t, x$, and $d_C(\{x\}, y') = k$, then $u_+(y') = (-1)^{k+1}\frac{c}{q^k}$. A similarly result holds for u_-. This result can be proved inductively, by considering first the edges of distance 1 from x: the sum on the positively oriented edges is 0, and since the q edges in $\mathcal{N}(a, x)$ which have x as an endpoint all have the same value, $u_+ = -\frac{c}{q}$. Repeating this process on all vertices of a given distance from x proves the result.

Furthermore, we can show the following using this result.

Theorem 2.4.3 *If $\tilde{u} \in \tilde{\mathcal{S}}$, then $\tilde{u} \in \ell^2(E_+ \cup E_-)$ and unless $\tilde{u} = 0$, $\tilde{u} \notin \ell^1(E_+ \cup E_-)$.*
This means that $u \in \mathcal{S}(y)$ is in $L^2(\mathrm{Aut}(\Gamma))$, but not in $L^1(\mathrm{Aut}(\Gamma))$ unless $u = 0$.

Proof:
We first note that

$$\sum_{y'} |u_+(y')|^2 = \sum_{y' \in B_y(n_u)} |u_+(y')|^2 + \sum_{y' \notin B_y(n_u)} |u_+(y')|^2$$

but $\sum_{y' \in B_y(n_u)} |u_+(y')|^2$ is a sum over a finite number of edges, and is hence finite. Now

$$\begin{aligned}
\sum_{y' \notin B_y(n_u)} |u_+(y')|^2 &= \sum_{x \in \partial B_y(n_u)} \sum_{k=1}^{\infty} \sum_{y' \in \mathcal{N}(a,x), d_C(\{x\},y')=k} |u_+(y')|^2 \\
&= \sum_{x \in \partial B_y(n_u)} \sum_{k=1}^{\infty} \sum_{y' \in \mathcal{N}(a,x), d_C(\{x\},y')=k} |\frac{c(x)}{q^k}|^2 \\
&= \sum_{x \in \partial B_y(n_u)} |c(x)|^2 \sum_{k=1}^{\infty} q^k \frac{1}{q^{2k}} \\
&= \sum_{x \in \partial B_y(n_u)} |c(x)|^2 \sum_{k=1}^{\infty} \frac{1}{q^k} \\
&= \sum_{x \in \partial B_y(n_u)} |c(x)|^2 \frac{q}{q-1} \\
&< \infty
\end{aligned}$$

Thus

$$\sum_{y'} |u_+(y')|^2 < \infty$$

a similar argument gives

$$\sum_{y'} |u_-(y')|^2 < \infty$$

and thus combining the two we get that $\tilde{u}$ is ℓ^2.

Assume that $\tilde{u} \neq 0$. Then there must be some $x \in \partial B_y(n_u)$ such that $u_+(\{t,x\}) \neq 0$ (or $u_-(\{t,x\}) \neq 0$; we will assume without loss of generality that it is u_+), since if it were, then $\tilde{u}$ would have to be 0 everywhere. Thus the ℓ^1 norm of $\tilde{u}$ must be at least the sum over the values of u_+ on the $y' \in \mathcal{N}(a,x)$, ie.

$$\begin{aligned}
\|\tilde{u}\|_1^2 &\geq \sum_{k=1}^{\infty} \sum_{y' \in \mathcal{N}(a,x), d_C(\{x\},y')=k} |u_+(y')| \\
&\geq \sum_{k=1}^{\infty} \sum_{y' \in \mathcal{N}(a,x), d_C(\{x\},y')=k} |\frac{c(x)}{q^k}| \\
&\geq \sum_{k=1}^{\infty} q^k \frac{|c(x)|}{q^k} \\
&\geq \sum_{k=1}^{\infty} |c(x)| \\
&= \infty
\end{aligned}$$

So therefore $\tilde{u}$ is ℓ^1 iff $\tilde{u} = 0$.

Finally we note that $u \in \mathcal{S}(y)$ has L^2 norm equal to some multiple of the ℓ^2 norm of the corresponding $\tilde{u}$, and similarly for L^1. (This multiple, as it turns out is $\mu(K_y) = \frac{1}{q+1}\mu(K_a) = \frac{1}{q+1}$). Thus the results carry directly over into $\mathcal{S}(y)$.

■

Proposition 2.4.4 *The subspace of $\mathcal{S}(y)$ of all left-K_y-invariant functions is a two dimensional subspace, and for every f in this subspace*

$$\|f\|_2^2 = \frac{1}{q-1}(|f_+(y)|^2 + |f_-(y)|^2)$$

Proof:

Let M be our left-K_y-invariant subspace. If $f \in M$, then $\tilde{f}$ is constant on edges of distance n from y. We note from above that this means that $\tilde{f}$ is determined completely by the values of f_+ and f_- on y.

More specifically, if $f_+(y) = c$, then $f_+(y') = c(\frac{-1}{q})^{d_E(y,y')}$, and if $f_-(y) = d$, then $f_+(y') = d(\frac{-1}{q})^{d_E(y,y')}$.

Let $\tilde{v}_1$ be the function which is 1 on the positively oriented component of y and 0 on E_-, and let $\tilde{v}_2$ be the function which is 0 on E_+ and 1 on the negatively oriented component of y. Then for any $f \in M$, $f = cv_1 + dv_2$. Thus $\{v_1, v_2\}$ is a complete basis for M.

Hence M is two dimensional.

Finally, from the above theorem, we can easily show that

$$\sum_{y' \in E} |f_+(y')|^2 = |f_+(y)|^2 \frac{q+1}{q-1}$$

and

$$\sum_{y' \in E} |f_-(y')|^2 = |f_-(y)|^2 \frac{q+1}{q-1}$$

Thus $\|f\|_2^2 = (|f_+(y)|^2 + |f_-(y)|^2)\frac{q+1}{q-1}\frac{1}{q+1} = \frac{1}{q-1}(|f_+(y)|^2 + |f_-(y)|^2)$ as required. ■

Note that in the context of $\mathcal{S}(y)$, this is saying that if u is in this subspace, then u is determined by its value on the set of automorphisms which stabilise y as a set.

So we have that $\mathcal{S}(y)$ is indeed a subset of $L^2(\mathrm{Aut}(\Gamma))$, and as it is at least 2-dimensional, it is non-trivial. Let $\mathcal{M}(y)$ be the closure of $\mathcal{S}(y)$ in $L^2(\mathrm{Aut}(\Gamma))$, so $\mathcal{M}(y)$ is a closed, non-trivial, invariant subspace of λ, (the left regular representation), harking back to the fact that $\mathcal{S}(y)$ is left-invariant.

Proposition 2.4.5 *$\mathcal{M}(y)$ can be identified with the subspace of $\ell^2(E_+ \cup E_-)$ such that for all $v \in V$, and $m \in \mathbf{N}$,*

$$\sum_{d_C(\{v\},y')=m} u_+(y') = 0$$

and

$$\sum_{d_C(\{v\},y')=m} u_+(y') = 0$$

(ie. property (ii) of $\tilde{\mathcal{S}}$).

Proof:

It suffices to check that all functions with this property are in the closure of $\mathcal{S}$.

Choose some $f \in \ell^2$ such that this holds. Then define the sequence $\{u_n\} \subset \mathcal{S}(y)$ by setting $\tilde{u} = \tilde{f}$ on $B_y(n)$ and letting property (i) of $\tilde{\mathcal{S}}$ hold for $\tilde{u}$ with $n_u = n$.

By similar arguments to the above Theorem and Lemma, it is possible to show that

$$\|f - u_n\|_2^2 \leq \mu(K_y)(1 + (q-1)^{-1/2})^2 (\sum_{y' \notin B_y(n)} (|f_+(y')|^2 + |f_-(y')|^2))$$

which tends to 0 as n goes to infinity.

Thus f is the limit of the u_n and we are done. ■

Let λ_y be the restriction of λ to $\mathcal{S}(y)$. We note that since $\rho(g)\mathcal{S}(y) = \mathcal{S}(gy)$, so therefore $\rho(g)\mathcal{M}(y) = \mathcal{M}(gy)$. Thus λ_y and λ_{gy} are unitarily equivalent.

Note that the bi-K_y-invariant elements of $\mathcal{M}(y)$ are exactly the bi-K_y-invariant elements of $\mathcal{S}(y)$, since if a function is both left- and right-K_y-invariant, it must be in $\mathcal{S}(y)$, by definition. Thus the subspace is still two dimensional.

In fact this space is exactly $P_\lambda(y)(\mathcal{M}(y))$. Indeed, we can exactly specify the action of $P_\lambda(y)$ on $\mathcal{M}(y)$, since, for $f \in \mathcal{M}(y)$, we have

$$\begin{aligned}
(P_\lambda(y)f)(e) &= \int_{K_y} \lambda(k)f(e)\; dk \\
&= \int_{K_y} f(k^{-1}e)\; dk \\
&= \int_{K_y} f(ek^{-1})\; dk \\
&= \int_{K_y} f(e)\; dk, \text{ since } f \in \mathcal{M}(y) \\
&= f(e)
\end{aligned}$$

and if g_0 is an inversion of y, then $g_0 y = y$, so we have

$$\begin{aligned}
(P_\lambda(y)f)(g_0) &= \int_{K_y} \lambda(k)f(g_0)\; dk \\
&= \int_{K_y} f(k^{-1}g_0)\; dk \\
&= \int_{K_{g_0 y}} f(g_0 h^{-1} g_0^{-1} g_0)\; dh \\
&= \int_{K_y} f(g_0 h^{-1})\; dh \\
&= \int_{K_y} f(g_0)\; dh, \text{ since } f \in \mathcal{M}(y) \\
&= f(g_0)
\end{aligned}$$

So the image of f under the action of $P_\lambda(y)$ is the function $u \in P_\lambda(y)(\mathcal{M}(y))$ such that $\tilde{u} = \tilde{f}$ on (a, b) and (b, a).

Lemma 2.4.6 *Every nontrivial, closed, left-invariant subspace of $\mathcal{M}(y)$ contains a nontrivial, left-K_y-invariant function from $\mathcal{M}(y)$.*

Proof:

Let M be our closed, non-trivial, left-invariant subspace of $\mathcal{M}(y)$.

Thus there is some $u \in M, u \neq 0$. We can assume without loss of generality that $u(e) \neq 0$ (since there must be some $g \in \text{Aut}(\Gamma))$ for which $u(g) \neq 0$, and so the left translate by g^{-1} must have non-zero value at e).

Now $P_\lambda(y)(M) \subseteq M$, since M is left-invariant, so $P_\lambda(y)u \in M$, but from above $(P_\lambda(y)u)(e) = u(e) \neq 0$, so $P_\lambda(y)u$ is a non-trivial left-K_y-invariant function in M. ■

This means that λ_y can have at most 2 irreducible subrepresentations, since if there were n subrepresentations, then there would be n closed, non-trivial, left-invariant subspaces, which means that the dimension of the left-K_y- invariant subspace would have to be $\geq n$. So there are at most 2.

Therefore either λ_y is irreducible, or $\lambda_y = \sigma_1 \oplus \sigma_2$, where σ_1, σ_2 are irreducible.

Lemma 2.4.7 *y is a minimal tree for λ_y and for any subrepresentations it might have.*

Proof:

For any $v \in V$, $f \in \mathcal{M}(y)$, $g \in \text{Aut}(\Gamma)$,

$$\begin{aligned}
(P_\lambda(v)f)(g) &= \int_{K_v} f(k^{-1}g)\, dk \\
&= \int_{K_{gv}} f(gk^{-1}g^{-1}g)\, dk \\
&= \int_{K_{gv}} f(gk^{-1})\, dk \\
&= 0
\end{aligned}$$

So $P_\lambda(v) = 0$ for all $v \in V$, and from above we have shown that there are functions for which $P_\lambda(y)$ is non-zero. Therefore y is minimal. ■

Thus the irreducible subrepresentations of λ_y are special.

Furthermore, they are essentially the only special representations, by the following lemma.

Lemma 2.4.8 *Every special representation is unitarily equivalent to a subrepresentation of λ_y.*

Proof:

If π is special, and $\xi \in P_\pi(y)\xi$, then $\langle \pi(g)\xi, \xi\rangle \in \mathcal{S}(y) \subseteq L^2(\text{Aut}(\Gamma))$. So π is square integrable, and so by the theory of square integrable representations π can be intertwined with λ, so $\pi \simeq$ some subrepresentation of λ_y (see Section 2.2). ■

Which brings us to our final theorem for this section.

Theorem 2.4.9 *λ_y is the direct sum of two inequivalent, irreducible subrepresentations σ_1 and σ_2.*

Proof:

Let $f_+(y) = f_-(y) = 1$ and $h_+ = 1, h_- = -1$ and $f, h \in$ the left- K_y-invariant subspace of $\mathcal{M}(y)$. This is a basis for this subspace, since $(1,1)$ and $(1,-1)$ are orthogonal in $\mathbf{C}^2$, so $\langle f, h\rangle = 0$ in $L^2(\text{Aut}(\Gamma))$.

Let g_0 be an inversion on y. Then $\lambda(g_0)f = f$ and $\lambda(g_o)h = -h$.

Now consider the closed subspaces M_1 and M_2 generated by the sets $\{\lambda(s)f : s \in \mathrm{Aut}(\Gamma)\}$ and $\{\lambda(s)h : s \in \mathrm{Aut}(\Gamma)\}$ respectively. These two spaces are not equal to one another, and are both clearly non-trivial closed subspaces of $\mathcal{M}(y)$.

Thus M_1 and M_2 carry subrepresentations of σ_1 and σ_2 of λ_y.

Let $F(t) = \langle \lambda(t)f, h\rangle$. F is left-K_y-invariant, since for all $k \in K_y$,

$$\begin{aligned} F(kt) &= \langle \lambda(kt)f, h\rangle \\ &= \langle \lambda(t)f, \lambda(k^{-1})h\rangle \\ &= \langle \lambda(t)f, h\rangle \end{aligned}$$

since h is K_y-invariant. However $F_+(y) = F_-(y) = \langle f, h\rangle = 0$, which implies that $F = 0$.

This gives us that $\langle v, w\rangle = 0$ for all $v \in M_1$, $w \in M_2$. Thus $M_1 \perp M_2$, and hence $M_1 \oplus M_2 = \mathcal{M}(y)$ (if there were any other direct summands, then they would have to be left-K_y-invariant, which is a contradiction).

Therefore $\lambda_y = \sigma_1 \oplus \sigma_2$, and they must both be irreducible.

Now assume that σ_1 is equivalent to σ_2. Then there is some unitary operator $U : M_1 \to M_2$ such that $U\sigma_1(g) = \sigma_2(g)U$. Since σ_1, σ_2 are subrepresentations of λ_y, so we can write $U\lambda_y(g) = \lambda_y(g)U$, which implies that $UP_\lambda(y) = P_\lambda(y)U$, that is U maps the left-K_y-invariant functions of M_1 to the left-K_y-invariant functions of M_2, which implies that $Uf = ch$, where $c = \pm 1$. However $\lambda(g_0)Uf = -ch$, but $U\lambda(g_0)f = Uf = ch$, which implies $U = 0$. Thus we have a contradiction. So the two representations are not equivalent. ■

It can also be shown that σ_1 and σ_2 have formal dimensions $d_{\sigma_1} = d_{\sigma_1} = \frac{q-1}{2}$ (see Figà-Talamanca [4, III, §2, p118]). This is useful in determining the Plancherel Formula for $L^2(\mathrm{Aut}(\Gamma))$ (see Section 2.6).

2.5 Cuspidal Representations

We now move on to briefly deal with the cuspidal representations of $\mathrm{Aut}(\Gamma)$. Let π be cuspidal, and let T be its minimal tree. We have the following initial results.

Lemma 2.5.1 *Let T be a complete subtree of Γ with diam$(T) \geq 2$. Let S be a finite complete subtree with $T \not\subseteq S$. Then there exists a proper complete subtree Z of T such that $K_Z \subseteq K_S K_T$.*

Proof:

Without loss of generality, we may assume that $T \cap S$ contains an edge. This follows since if $T \cap S$ is either empty or a single edge, then there exists some unique $v \in T$ of minimal distance m from S. We then set $S' = B_S(m+1) = S_{m+1}$. The intersection of T and S' is clearly an edge, and if $T \subset S'$ then either T has diameter < 2 or T is not complete. Clearly $K_{S'} \subset K_S$.

Let $Z = S \cap T$. Then Z is a complete, proper subtree of T. It is also a complete subtree of S. Hence given any $k \in K_Z$ we can find an automorphism $k_2 \in K_T$ which agrees with k on $S \setminus T$. Similarly, we can find an automorphism $k_1 \in K_S$ such that $k_1 = kk_2^{-1}$, since kk_2^{-1} clearly stabilises S. Hence $k = k_1 k_2$ and therefore $K_Z \subseteq K_S K_T$. ■

Proposition 2.5.2 *Let T be a finite complete subtree of Γ with diam$(T) \geq 2$. Let S be a finite complete subtree of Γ. If u is a left-K_S-invariant element of $\mathcal{S}(T)$, then supp$(u) \subset \{g \in \mathrm{Aut}(\Gamma) : gT \subseteq S\}$.*

Proof:

We want to show that if $u \in \mathcal{S}(T)$ is left-K_S-invariant, then $u(g) = 0$ for all g such that $gT \nsubseteq S$.

If $g = e$ and $T \nsubseteq S$, then there exists a complete $Z \subset T$ such that $K_Z \subseteq K_S K_T$. Since u is left-K_S-invariant and right-K_T-invariant, then u must be constant on K_Z. However, since $Z \subset T$, we have that for all $h \in K_Z$,

$$\int_{K_Z} u(gk)\ dk = \int_{K_Z} u(g)\ dk = 0$$

which means that $u(h) = 0$ for all $h \in K_Z$. In particular, this means $u(e) = 0$.

Now, for more general g, we consider $u(kg) \in \mathcal{S}(gT)$. $u(kg)$ is still left-K_S-invariant, so by the above argument, if $gT \nsubseteq S$, then $u(eg) = u(g) = 0$. ∎

Corollary 2.5.3 *Let $[T] = \{gT : g \in \mathrm{Aut}(\Gamma)\}$, then T' is minimal for π iff $T' \in [T]$.*

Proof:

We already know that if T is minimal, then gT is also minimal.

If T' is minimal, then T' must have ℓ_π vertices. Furthermore, let ξ be a non-trivial K_T-invariant vector, and let η be a non-trivial $K_{T'}$-invariant vector. Then $\langle \pi(.)\xi, \eta \rangle \in \mathcal{S}(T)$ where $T_{\langle \pi(.)\xi,\eta \rangle} = T'$. This function is non-trivial, and so has non-trivial support. Hence by the Proposition above, there is some g such that $gT \subseteq T'$. Therefore $gT = T'$, since the number of vertices in T, gT and T' are the same. ∎

Corollary 2.5.4 $\mathcal{S}(T) \subseteq L^2(\mathrm{Aut}(\Gamma))$

Proof:

If $u \in \mathcal{S}(T)$, then $|u| < \infty$, since if there were some $g \in \mathrm{Aut}(\Gamma)$ such that $u(g) = \infty$, then $u(gk) = \infty$ for all $k \in K_T$, which implies since $K_T \subset K_{T^\circ}$ for a complete proper subtree T° of T, and $\mu(K_T) > 0$, that

$$\int_{K_{T^\circ}} u(gk)\ dk = \infty$$

Hence, since T has compact support, and u is bounded, it must be L^2. ∎

We are again interested in the subspace of left-K_T-invariant functions. As with the previous section, we can show that the function is determined by its value on the subgroup of automorphisms which stabilise T as a set, ie. the subgroup $\tilde{K}_T = \{g \in \mathrm{Aut}(\Gamma) : gT = T\}$. It can be shown that K_T is normal in $\tilde{K}_T$, and since these functions are constant on K_T it makes sense to consider them as functions on $\tilde{K}_T/K_T$, which is a finite group isomorphic to $\mathrm{Aut}(T)$.

An argument similar to the one in the previous section will give us that the subspace of left-K_T-invariant functions has dimension equal to the number of elements in $\tilde{K}_T/K_T$.

We can therefore consider $\mathcal{S}(T)$ as the space of functions on $\Gamma \times \mathrm{Aut}(\Gamma)$. This space is the analogue of $\tilde{\mathcal{S}}$ in the special case, and so we will call it $\tilde{\mathcal{S}}(T)$

Let $\mathcal{M}(T)$ be the subspace of $L^2(\Gamma)$ consisting of functions for which conditions (i) and (iii) of $\mathcal{S}(T)$ hold.

Clearly we have that $\mathcal{S}(T) \subset \mathcal{M}(T)$, and $\mathcal{M}(T)$ is a non-trivial, closed, left-invariant subspace of L^2. Thus we can restrict λ to $\mathcal{M}(T)$, and we will call this restriction λ_T. Again, λ_T is unitarily equivalent to λ_{gT}. Finally $P_\lambda(T')\mathcal{M}(T) \subset \mathcal{S}(T)$ for all finite complete subtrees T'.

Proposition 2.5.5 *$\mathcal{M}(T)$ is the closure of $\mathcal{S}(T)$ in $L^2(\mathrm{Aut}(\Gamma))$.*

Proof:

The proof of this involves showing that any $f \in \mathcal{M}(T)$ is the limit of some sequence f_n in $\mathcal{S}(T)$.

The proof of this is essentially the same as for the special case, but we use $\tilde{\mathcal{S}}(T)$ instead of $\tilde{\mathcal{S}}$ (see Proposition 2.4.5). ∎

We can again deduce the action of $P_\lambda(T)$, since if $g \in \tilde{K}_T$, then

$$\begin{aligned}
(P_\lambda(T)f)(g) &= \int_{K_T} \lambda(k)f(g)\ dk \\
&= \int_{K_T} f(k^{-1}g)\ dk \\
&= \int_{K_{gT}} f(gh^{-1}g^{-1}g)\ dh \\
&= \int_{K_T} f(gh^{-1})\ dh \\
&= \int_{K_T} f(g)\ dh, \text{ since } f \in \mathcal{M}(T) \\
&= f(g)
\end{aligned}$$

Thus the action of $P_\lambda(T)$ is to take f to the left-K_T-invariant function which agrees with it on $\tilde{K}_T$.

We now have, by analogy with the special case, the following results

Lemma 2.5.6 *Every nontrivial, closed, left-invariant subspace of $\mathcal{M}(T)$ contains a non-trivial, left-K_T-invariant function from $\mathcal{M}(T)$.*

Lemma 2.5.7 *T is a minimal tree for λ_T and for any subrepresentations it might have.*

Lemma 2.5.8 *Every special representation is unitarily equivalent to a subrepresentation of λ_T.*

Proof: These are proved in exactly the same fashion as for the special case. (See Lemmas 2.4.6,2.4.7 and 2.4.8).

Before we can identify the irreducible subrepresentations, we need to introduce the following.

Definition 2.5.1 *Let T_i, $i = 1, \ldots, j$ be the maximal complete proper subtrees of T.*

A unitary representation of $\tilde{K}_T$ is called non-degenerate *if it has no nonzero K_{T_i}-invariant vectors for $i = 1, \ldots, j$.*

Let $(\tilde{K}_T)^\wedge_0$ be the set of all such representations which are trivial on K_T.

Then we have the following.

Theorem 2.5.9 *If*

$$\sigma_T = \bigoplus_{\sigma \in (\tilde{K}_T)^\wedge_0} (dim\sigma)\sigma$$

then λ_T is unitarily equivalent to

$$Ind(\sigma_T) = \bigoplus_{\sigma \in (\tilde{K}_T)^\wedge_0} (dim\sigma) Ind(\sigma)$$

and each of the $Ind(\sigma)$ is irreducible (and therefore cuspidal).

Proof: See Theorem 3.14 of Figà-Talamanca [4, III, §3, p134].

Furthermore, it is also possible to show that for each finite complete subtree T of diameter ≥ 2, there exists a cuspidal representation π for which T is a minimal tree. (See Figà-Talamanca [4, III, §3, p124]). This is important in finding the Plancherel Formula for $L^2(\mathrm{Aut}(\Gamma))$.

2.6 Extensions and Generalisations

In this section we will tie up some loose ends.

Spherical Representations

Although spherical representations cannot be dealt with in this thesis, as the required theory would add another 30 pages, the basic method of attack and the main results should be given. For complete details, the interested reader is directed to Chapter II of Figà-Talamanca [4], and the papers by Cartier [1] and Figà-Talamanca and Steger [5].

The approach is via investigation of *spherical harmonic functions* on Γ, that is functions f which depend only on the distance from some fixed vertex $o \in V$, and in addition have the property that

$$Lf(u) = (q+1)^{-1} \sum_{v \in \partial B_o(1)} f(u) = cf(u)$$

for some constant c (L is called the *Laplace operator* on Γ, and hence f is an eigenfunction of L), and finally $f(o) = 1$.

It can be shown that the spherical representations of $\mathrm{Aut}(\Gamma)$ correspond to the positive definite spherical functions, and that they have the form $\pi_z : G \to L^2(\Omega, \nu_o)$ where $\langle \pi_z 1, 1\rangle$ is the positive definite spherical function with associated eigenvalue $(q+1)^{-1} q^{1/2} 2\mathrm{Re}(q^{it})$ where $z = \frac{1}{2} + it$ for $t \in [0, \pi/\ln(q)]$, and 1 is the function which is 1 everywhere on Ω.

From this it is possible to get that for all left-K_o-invariant functions f on $\mathrm{Aut}\Gamma$ with compact support, we have

$$\|f\|^2 = \int_J \|\pi_{\frac{1}{2}+it}(f)\|^2_{HS}\, dm(t)$$

where $J = [0, \pi/\ln(q)]$ and $dm(t) = \frac{q\ln(q)}{2\pi(q+1)} |c(\frac{1}{2}+it)|^{-2} dt$, where $c(z) = \frac{1}{q+1} \frac{q^{1-z} - q^{z-1}}{q^{-z} - q^{z-1}}$, and $\|.\|_{HS}$ is the Hilbert-Schmidt norm defined on the bounded linear operators on the appropriate Hilbert space, where $\|T\|_{HS} = \mathrm{Tr}(T^*T)^{1/2}$.

The Plancherel Formula for $L^2(\mathrm{Aut}(\Gamma))$

Again we will just state the basic result, which is that for every continuous function f on $\mathrm{Aut}(\Gamma)$ with compact support, we have:

$$\begin{aligned} \|f\|_2^2 &= \int_J \|\pi_{\frac{1}{2}+it}(f)\|_{HS}^2 \; dm(t) + \frac{q-1}{2} \sum_{i=1} 2(\mathrm{Tr}(\sigma_i(f)\sigma_i(f)^*) \\ &+ \sum_{[T],\mathrm{diam}(T)\geq 2} \frac{1}{\mu(\tilde{K}_T)} (\sum_{\sigma \in (\tilde{K}_T)_0^\wedge} \dim \sigma \mathrm{Tr}(\mathrm{Ind}(\sigma)(f)(\mathrm{Ind}(\sigma)(f))^*)) \end{aligned}$$

For the reasoning behind this, see Figà-Talamanca [4, III, §3, p136].

Other Groups

Much of this chapter holds for any closed subgroup G of $\mathrm{Aut}(\Gamma)$ which can be written in the form $G = KF = FK$ where K is compact and F is a transitive subgroup for which $g^{-1}Kg \cap F = \{e\}$, since then G is unimodular, and it will have most of the required properties. However, it also needs to have a sufficiently large rotation group for the section on cuspidal representations to hold.

The two most interesting such subgroups would be $\mathrm{Aut}_+(\Gamma)$ which is the subgroup of $\mathrm{Aut}(\Gamma)$ generated by all rotations (ie. it is inversion-free), and $PSL(2, P)$, where P is a p-adic field (which acts on a homogeneous tree, see Figà-Talamanca [4, Appendix, §5, p156]. Both these groups are inversion free which means that there is only one irreducible special representation. In the case of $\mathrm{Aut}_+(\Gamma)$ the method for the cuspidal representations still works, however, $PSL(2, P)$ has insufficient rotations.

Again, for more information, see Chapter III of Figà-Talamanca [4].

Bibliography

[1] P. Cartier, *Harmonic Analysis on Trees*, Proc. Symp. Pure Math. A.M.S., 26 (1972), pp419–424.

[2] J. Dixmier[1], *Les C^*-algebras et Leurs Representations*, Gauthier-Villars, Paris, 1964.

[3] R.E. Edwards, *Integration and Harmonic Analysis on Compact Groups*, Australian National University, Canberra, 1970.

[4] A. Figà-Talamanca and C. Nebbia, *Harmonic Analysis and Representation Theory for Groups Acting on Homogeneous Trees*, Manuscript.

[5] A. Figà-Talamanca and T. Steger, *Harmonic Analysis on Trees*, Symp. Math., 29 (1987), pp163–182.

[6] P.R. Halmos, *Measure Theory*, Van Nostrand, Princeton, 1950.

[7] L.H. Loomis, *An Introduction to Abstract Harmonic Analysis*, Van Nostrand, Princeton, 1953.

[8] G.I. Ol'shianskii, *Classificiation of Irreducible Representations of Groups of Automorphisms of Bruhat-Tits Trees*, Functional Anal. Appl., (1977), pp26–34.

[9] H. Reiter, *Classical Harmonic Analysis and Locally Compact Groups*, Oxford University Press, Oxford, 1968.

[10] D.J.S. Robinson, *A Course in the Theory of Groups*, Springer-Verlag, New York, 1982.

[11] J.P. Serre, *Trees*, Springer-Verlag, New York and Berlin, 1980.

[1] Appropriate sections translated by Professor Sutherland